

Basis Sine and Cosine (Quaternion Sine and Cosine) Identities

Principal basis operator sines and cosines (basis sines and cosines: **sink**, **sinj**, **sini**, **cosk**, **cosj**, **cosi**, **tank**, **tanj**, and **tani**) can also be called **principal quaternion basis operator** sines and cosines (quaternion sines and cosines). I have not adopted this nomenclature earlier because this terminology may be misconstrued as the scalar sines and cosines ($\sin$, $\cos$, and $\tan$) on the basis variables after the basis operators have operated on the basis and converted to scalars. It is the explicit dependence on $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$ that give these functions their complex-number-like and quaternion-like properties (despite there being no such thing as complex numbers... it is all vector operators in the real R^3 vector space). **Quaternion basis exponentials** is also an appropriate terminology for basis exponentials **expk**, **expj**, and **expi** (**cisk**, **cisj**, and **cisi**).

Trigonometric Identities of **sink** and **cosk**

All of the periodic properties of the scalar trigonometric functions and identities of sine and cosine (and tangent) hold good for basis sines and cosines (and tangents). This document focuses on those of **sink** and **cosk**, and then those, by change of base, of **sinj** and **cosj** and similarly **sini** and **cosi**.

I will give the proofs here for the main identities I required for the Law of Basis Sines, which is after this (and is why I wrote this part which has turned out quite well). I'll prove these formulae of **sink** and **cosk** with **expk** and then change base with (1) **expj**($-\pi/2$) and **expj**($\pi/2$), and (2) **expi**($\pi/2$) and **expi**($-\pi/2$) respectively, to (1) **sinj** and **cosj**, and (2) **sini** and **cosi**. All of the proofs for **cosk** and **sink** are with the basis operators in terms of the Euler formulae for those operators

$$\begin{aligned} \text{cosk}\theta &= \frac{1}{2}(\text{expk}(\theta) + \text{expk}(-\theta)) \\ &= \frac{1}{2}(\exp(\theta \hat{k} \times) + \exp(-\theta \hat{k} \times)) \\ &= \frac{1}{2}(e^{\theta \hat{k} \times} + e^{-\theta \hat{k} \times}) \end{aligned} \quad \begin{aligned} \text{sink}\theta \hat{k} \times &= \frac{1}{2}(\text{expk}(\theta) - \text{expk}(-\theta)) \\ &= \frac{1}{2}(\exp(\theta \hat{k} \times) - \exp(-\theta \hat{k} \times)) \\ &= \frac{1}{2}(e^{\theta \hat{k} \times} - e^{-\theta \hat{k} \times}) \end{aligned}$$

These are then proved again with the diadically expanded matrix operators which essentially proves further that it all works in matrix bases. These can all be verified easily with a programmable calculator (which I do) to check that the formulae are correct. These would also be very good functions for MATLAB base code (which is very similarly code to my programmable calculator). Alternatively, and equivalently, to change base to the other principal rotation axes, all of the relations can be proved for the other rotation axes with

$$\begin{aligned} \text{cosj}\theta &= \frac{1}{2}(\text{expj}(\theta) + \text{expj}(-\theta)) \\ &= \frac{1}{2}(\exp(\theta \hat{j} \times) + \exp(-\theta \hat{j} \times)) \\ &= \frac{1}{2}(e^{\theta \hat{j} \times} + e^{-\theta \hat{j} \times}) \end{aligned} \quad \begin{aligned} \text{sinj}\theta \hat{j} \times &= \frac{1}{2}(\text{expj}(\theta) - \text{expj}(-\theta)) \\ &= \frac{1}{2}(\exp(\theta \hat{j} \times) - \exp(-\theta \hat{j} \times)) \\ &= \frac{1}{2}(e^{\theta \hat{j} \times} - e^{-\theta \hat{j} \times}) \end{aligned}$$

and

$$\begin{aligned} \text{cosi}\theta &= \frac{1}{2}(\text{expi}(\theta) + \text{expi}(-\theta)) \\ &= \frac{1}{2}(\exp(\theta \hat{i} \times) + \exp(-\theta \hat{i} \times)) \\ &= \frac{1}{2}(e^{\theta \hat{i} \times} + e^{-\theta \hat{i} \times}) \end{aligned} \quad \begin{aligned} \text{sini}\theta \hat{i} \times &= \frac{1}{2}(\text{expi}(\theta) - \text{expi}(-\theta)) \\ &= \frac{1}{2}(\exp(\theta \hat{i} \times) - \exp(-\theta \hat{i} \times)) \\ &= \frac{1}{2}(e^{\theta \hat{i} \times} - e^{-\theta \hat{i} \times}) \end{aligned}$$

The formulae for the basis sines (**sink**, **sinj**, and **sini**) are appended by the relevant principal basis cross ($\hat{k} \times$, $\hat{j} \times$, or $\hat{i} \times$) to avoid the issue of singular (non-full rank) $\hat{k}$ -cross, $\hat{j}$ -cross, and $\hat{i}$ -cross in the denominator. It all works out well when defined as I have done so here.

The hyperbolic identities with **sinkk** and **coskk** (**k-cross squared equals negative one**) could equivalently be called **sinhk** and **coshk** (“shine-kay” and “cosh-kay” and **sinhj** “shine-jay” and **coshj** “cosh-jay” and **sinih** “shine-eye” and **coshi** “cosh-eye”) but I prefer **sinkk** (“sine-kay-kay”) and **coskk** (“cos-kay-kay”) because it stresses the quintessential feature of complex numbers $(\hat{k} \times)^2 = -1$ which is also the feature that provides the hyperbolics with either divergent or convergent properties, notably, that of reflection. It is also better because the square of $\hat{k} \times$ results in a sign change (which is the reflection). Similarly with $(\hat{j} \times)^2 = -1$ and $(\hat{i} \times)^2 = -1$ and why I prefer the notation and terminology **sinjj** (“sine-jay-jay”) and **cosjj** (“cos-jay-jay”) and **sinih** (“sine-eye-eye”) and **cosih** (“cos-eye-eye”).

Note that a conversion from **sink** and **cosk** to **sinkk** and **coskk** is to make the argument of **sink** and **cosk** a function of $k \times$, similarly as for 2nd year complex analysis and calculus converting sines and cosines to and from the hyperbolic functions. This transform makes the complex formulae relating circular and hyperbolic functions in a vector setting possible as well as simple and intuitive, because of that important fact that **sinkk** and **coskk** get their hyperbolic properties from reflection from the square of $i = \hat{k} \times$. It all works as it should for and with $i = \hat{k} \times$, including for the other R^3 mutually orthogonal planes with $\hat{j} \times$ and $\hat{i} \times$. (since these are both similarity invariants of $i = \hat{k} \times$ after a principal axis base change).

Periodic properties of **cosk** and **sink** (and **tank**)

Firstly, angle reversal for the basis circular functions are the same as the scalar functions $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$, and $\tan(-\theta) = -\tan\theta$. As is seen in the proofs, this holds independently of the rotation axis

$$\text{cosk}(-\theta) = \text{cosk}\theta$$

$$\text{sink}(-\theta) \hat{k} \times = -\text{sink}\theta \hat{k} \times$$

$$\text{tank}(-\theta) \hat{k} \times = -\text{tank}\theta \hat{k} \times$$

that is,

scalar	k -hat rotation axis	j -hat rotation axis	i -hat rotation axis
$\cos(-\theta) = \cos\theta$	$\text{cosk}(-\theta) = \text{cosk}\theta$	$\text{cosj}(-\theta) = \text{cosj}\theta$	$\text{cosi}(-\theta) = \text{cosi}\theta$
$\sin(-\theta) = -\sin\theta$	$\text{sink}(-\theta) \hat{k} \times = -\text{sink}\theta \hat{k} \times$	$\text{sinj}(-\theta) \hat{j} \times = -\text{sinj}\theta \hat{j} \times$	$\text{sini}(-\theta) \hat{i} \times = -\text{sini}\theta \hat{i} \times$
$\tan(-\theta) = -\tan\theta$	$\text{tank}(-\theta) \hat{k} \times = -\text{tank}\theta \hat{k} \times$	$\text{tanj}(-\theta) \hat{j} \times = -\text{tanj}\theta \hat{j} \times$	$\text{tani}(-\theta) \hat{i} \times = -\text{tani}\theta \hat{i} \times$

These proofs are essentially properties of complex algebra and Euler's identities since all of complex algebra, including phasors, holds for $i = \hat{k} \times$ as well as for the two other axes (and those of r -hat, θ -hat, and ϕ -hat) and so generalize into vector functions in R^3 . The proofs are then also expressed in matrix bases:

$$\text{cosk}(-\theta) = \frac{1}{2} \left(e^{(-\theta)\hat{k} \times} + e^{-(-\theta)\hat{k} \times} \right) = \frac{1}{2} \left(e^{-\theta\hat{k} \times} + e^{\theta\hat{k} \times} \right) = \frac{1}{2} \left(e^{\theta\hat{k} \times} + e^{-\theta\hat{k} \times} \right) = \text{cosk}\theta$$

$$\text{cosk}(-\theta) = \begin{pmatrix} \cos(-\theta) & 0 & 0 \\ 0 & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{cosk}\theta$$

$$\text{sink}(-\theta) \hat{k} \times = \frac{1}{2} \left(e^{(-\theta)\hat{k} \times} - e^{-(-\theta)\hat{k} \times} \right) = \frac{1}{2} \left(e^{-\theta\hat{k} \times} - e^{\theta\hat{k} \times} \right) = -\frac{1}{2} \left(e^{\theta\hat{k} \times} - e^{-\theta\hat{k} \times} \right) = -\text{sink}\theta \hat{k} \times$$

$$\text{sink}(-\theta) \hat{k} \times = \begin{pmatrix} 0 & -\sin(-\theta) & 0 \\ \sin(-\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sin\theta & 0 \\ -\sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\text{sink}\theta \hat{k} \times$$

$$\text{tank}(-\theta) \hat{k} \times = (\text{cosk}(-\theta))^{-1} \text{sink}(-\theta) \hat{k} \times = (\text{cosk}\theta)^{-1} \text{sink}\theta \hat{k} \times = \text{tank}\theta \hat{k} \times$$

$$\text{tank}(-\theta) \hat{k} \times = \begin{pmatrix} 0 & -\tan(-\theta) & 0 \\ \tan(-\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tan\theta & 0 \\ -\tan\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -\tan\theta & 0 \\ \tan\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\text{tank}\theta \hat{k} \times$$

The scalar properties of $\cos(\theta + 2\pi n) = \cos\theta$, $\sin(\theta + 2\pi n) = \sin\theta$, and $\tan(\theta + 2\pi n) = \tan\theta$ are also essentially the same for **cosk**, **sink**, and **tank** and also hold for each coordinate axis. For $n = 0, \pm 1, \pm 2, \pm 3, \dots$ these are

$$\text{cosk}(\theta + 2\pi n) = \text{cosk}\theta$$

$$\text{sink}(\theta + 2\pi n) \hat{k} \times = \text{sink}\theta \hat{k} \times$$

$$\text{tank}(\theta + 2\pi n) \hat{k} \times = \text{tank}\theta \hat{k} \times$$

that is,

scalar	k -hat rotation axis	j -hat rotation axis	i -hat rotation axis
$\cos(\theta + 2\pi n) = \cos\theta$	$\mathbf{cosk}(\theta + 2\pi n) = \mathbf{cosk}\theta$	$\mathbf{cosj}(\theta + 2\pi n) = \mathbf{cosj}\theta$	$\mathbf{cosi}(\theta + 2\pi n) = \mathbf{cosi}\theta$
$\sin(\theta + 2\pi n) = \sin\theta$	$\mathbf{sink}(\theta + 2\pi n) \hat{k} \times = \mathbf{sink}\theta \hat{k} \times$	$\mathbf{sinj}(\theta + 2\pi n) \hat{j} \times = \mathbf{sinj}\theta \hat{j} \times$	$\mathbf{sini}(\theta + 2\pi n) \hat{i} \times = \mathbf{sini}\theta \hat{i} \times$
$\tan(\theta + 2\pi n) = \tan\theta$	$\mathbf{tank}(\theta + 2\pi n) \hat{k} \times = \mathbf{tank}\theta \hat{k} \times$	$\mathbf{tanj}(\theta + 2\pi n) \hat{j} \times = \mathbf{tanj}\theta \hat{j} \times$	$\mathbf{tani}(\theta + 2\pi n) \hat{i} \times = \mathbf{tani}\theta \hat{i} \times$

Proof for the relation $\mathbf{cosk}(\theta + 2\pi n) = \mathbf{cosk}\theta$ for $\mathbf{cosk}$ and $n = 0, \pm 1, \pm 2, \pm 3, \dots$

$$\mathbf{cosk}(\theta + 2\pi n) = \frac{1}{2} \left(e^{(\theta + 2\pi n) \hat{k} \times} + e^{-(\theta + 2\pi n) \hat{k} \times} \right) = \frac{1}{2} \left(e^{\theta \hat{k} \times} e^{2\pi n \hat{k} \times} + e^{-\theta \hat{k} \times} e^{-2\pi n \hat{k} \times} \right) = \frac{1}{2} \left(e^{\theta \hat{k} \times} + e^{-\theta \hat{k} \times} \right) = \mathbf{cosk}\theta$$

since $e^{2\pi n \hat{k} \times} = e^{-2\pi n \hat{k} \times} = I$ or in other words

$$\begin{pmatrix} \cos 2\pi n & -\sin 2\pi n & 0 \\ \sin 2\pi n & \cos 2\pi n & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos 2\pi n & \sin 2\pi n & 0 \\ -\sin 2\pi n & \cos 2\pi n & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

as diadically expanded matrix operators

$$\mathbf{cosk}(\theta + 2\pi n) = \begin{pmatrix} \cos(\theta + 2\pi n) & 0 & 0 \\ 0 & \cos(\theta + 2\pi n) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{cosk}\theta$$

Proof for the relation $\mathbf{sink}(\theta + 2\pi n) \hat{k} \times = \mathbf{sink}\theta \hat{k} \times$ for $\mathbf{sink}$ and $n = 0, \pm 1, \pm 2, \pm 3, \dots$

$$\mathbf{sink}(\theta + 2\pi n) \hat{k} \times = \frac{1}{2} \left(e^{(\theta + 2\pi n) \hat{k} \times} - e^{-(\theta + 2\pi n) \hat{k} \times} \right) = \frac{1}{2} \left(e^{\theta \hat{k} \times} e^{2\pi n \hat{k} \times} - e^{-\theta \hat{k} \times} e^{-2\pi n \hat{k} \times} \right) = \frac{1}{2} \left(e^{\theta \hat{k} \times} - e^{-\theta \hat{k} \times} \right) = \mathbf{sink}\theta \hat{k} \times$$

as diadically expanded matrix operators

$$\mathbf{sink}(\theta + 2\pi n) \hat{k} \times = \begin{pmatrix} 0 & -\sin(\theta + 2\pi n) & 0 \\ \sin(\theta + 2\pi n) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{sink}\theta \hat{k} \times$$

Proof similarly for $\mathbf{tank}(\theta + 2\pi n) \hat{k} \times = \mathbf{tank}\theta \hat{k} \times$

$$\mathbf{tank}(\theta + 2\pi n) \hat{k} \times = \begin{pmatrix} 0 & -\tan(\theta + 2\pi n) & 0 \\ \tan(\theta + 2\pi n) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\tan\theta & 0 \\ \tan\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{tank}\theta \hat{k} \times$$

Proofs are similar, as similarity invariants, for the other two principal axes and as well as the rotated polar cylindrical and spherical coordinate systems.

Who is the first mathematician to use the abbreviations sin cos and tan?

Albert Girard

Albert Girard was a French mathematician and musician. His work on trigonometry is the first to use the abbreviations sin, cos, tan.



st-andrews.ac.uk

<https://mathshistory.st-andrews.ac.uk> > Biographies > Gir...

Albert Girard (1595 - 1632) - Biography

I read an essay on trigonometry in Clayton Dodge's book *The Circular Functions* (chapter 6) and apparently the first simplified notation of sin, cos, and tan was in the 1600s by Albert Girard... so I included the screenshot above. Also of relevance:

Hipparchus

The first trigonometric table was apparently compiled by **Hipparchus**, who is consequently now known as "the father of trigonometry".



wikipedia.org

<https://en.wikipedia.org/wiki/Hipparchus>

Hipparchus - Wikipedia



The trigonometric relations (from the ancient Athenians and subsequent mathematicians) for scalar sines, cosines, and tangents (sin, cos, and tan) are

(A) Circular Euler form $\text{cis} = \exp$

$$\cos \theta + i \sin \theta = e^{i\theta}$$

(B) Circular Conjugate Euler form

$$\cos \theta - i \sin \theta = e^{-i\theta}$$

(1) Basic circular relation for sine and cosine

$$\cos^2 \theta + \sin^2 \theta = 1$$

(2) Basic circular relation for tangent and secant

$$1 + \tan^2 \theta = \sec^2 \theta$$

(3) Compound angle formula for sine

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

(4) Compound angle formula for cosine

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

(5) Compound angle formula for tangent

$$\tan(\alpha + \beta) = (\tan \alpha + \tan \beta)(1 - \tan \alpha \tan \beta)^{-1}$$

(6) Double angle formula for sine

$$\sin(2\alpha) = 2 \sin \alpha \cos \alpha$$

(7) Double angle formula for cosine

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha$$

(8) Double angle formula for tangent

$$\tan(2\alpha) = 2 \tan \alpha (1 - \tan^2 \alpha)^{-1}$$

(9) Products of sines

$$2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

(10) Products of cosines

$$2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$$

(11) Products of sine and cosine

$$2 \sin \alpha \cos \beta = \sin(\alpha - \beta) + \sin(\alpha + \beta)$$

The relations for xy plane ($\hat{k}\times$ rotation axis) basis sines, cosines, and tangents (**sink**, **cosk**, and **tank**) are

(A) Circular Euler form for cisk = expk	$\text{cosk}\theta + \text{sink}\theta \hat{k}\times = \text{expk}\theta$
(B) Circular conjugate Euler form	$\text{cosk}\theta - \text{sink}\theta \hat{k}\times = \text{expk}(-\theta)$
(1) Basic circular relation for sink and cosk	$\text{cosk}^2\theta - (\text{sink}\theta \hat{k}\times)^2 = 1$
(2) Basic circular relation for tank and seck	$1 - (\text{tank}\theta \hat{k}\times)^2 = \text{seck}^2\theta$
(3) Compound angle formula for sink	$\text{sink}(\alpha + \beta) \hat{k}\times = \text{cosk}\alpha \text{sink}\beta \hat{k}\times + \text{cosk}\beta \text{sink}\alpha \hat{k}\times$
(4) Compound angle formula for cosk	$\text{cosk}(\alpha + \beta) = \text{cosk}\alpha \text{cosk}\beta + (\text{sink}\alpha \hat{k}\times)(\text{sink}\beta \hat{k}\times)$
(5) Compound angle formula for tank	$\text{tank}(\alpha + \beta) \hat{k}\times = \left(\text{tank}\alpha \hat{k}\times + \text{tank}\beta \hat{k}\times \right) \left(1 - (\text{tank}\alpha \hat{k}\times)(\text{tank}\beta \hat{k}\times) \right)^{-1}$
(6) Double angle formula for sink	$\text{sink}(2\alpha) \hat{k}\times = 2\text{cosk}\alpha \text{sink}\alpha \hat{k}\times$
(7) Double angle formula for cosk	$\text{cosk}(2\alpha) = \text{cosk}^2\alpha + (\text{sink}\alpha \hat{k}\times)^2$
(8) Double angle formula for tank	$\text{tank}(2\alpha) \hat{k}\times = \left(2\text{tank}\alpha \hat{k}\times \right) \left(1 - (\text{tank}\alpha \hat{k}\times)^2 \right)^{-1}$
(9) Products of sink	$2(\text{sink}\alpha \hat{k}\times)(\text{sink}\beta \hat{k}\times) = \text{cosk}(\alpha - \beta) - \text{cosk}(\alpha + \beta)$
(10) Products of cosk	$2\text{cosk}\alpha \text{cosk}\beta = \text{cosk}(\alpha - \beta) + \text{cosk}(\alpha + \beta)$
(11) Products of sink and cosk	$2(\text{sink}\alpha \hat{k}\times) \text{cosk}\beta = \text{sink}(\alpha - \beta) \hat{k}\times + \text{sink}(\alpha + \beta) \hat{k}\times$

Notice how $\text{cosk}^2\theta - (\text{sink}\theta \hat{k}\times)^2 = 1$ looks kind of like $i^2 = -1$? That is no coincidence because this equation contains $(\hat{k}\times)^2 = -1$ coupled with the square of **sink** which, yeah, makes it work out. The visual appeal of complex algebra coupled with its practicality makes it, in my opinion, mathematical beauty at its best. Similarly (by congruence and change of base) with $\text{cosj}^2\theta - (\text{sinj}\theta \hat{j}\times)^2 = 1$ and $\text{cosi}^2\theta - (\text{sini}\theta \hat{i}\times)^2 = 1$ that contain $(\hat{j}\times)^2 = -1$ and $(\hat{i}\times)^2 = -1$ respectively, which makes it very clear that principal quaternion basis sines and cosines is good possible nomenclature for these functions.

Diadically expanded into matrix bases:

Noting again for reference that I have coupled **sink** with $\hat{k}\times$

$$\text{sink}\theta = \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \quad \text{sink}\theta \hat{k}\times = \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(A) Circular Euler form for **cisk** = **expk**

$$\cos\theta + i\sin\theta = e^{i\theta} \quad \text{cosk}\theta + \text{sink}\theta \hat{k}\times = \text{expk}\theta$$

$$\begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(B) Circular Conjugate Euler form

$$\cos\theta - i\sin\theta = e^{-i\theta} \quad \text{cosk}\theta - \text{sink}\theta \hat{k}\times = \text{expk}(-\theta)$$

$$\begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) Basic circular relation for sink and cosk

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\mathbf{cosk}^2\theta - (\mathbf{sink}\theta \hat{k} \times)^2 = 1$$

$$\begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Basic circular relation for tank and seck

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 - (\mathbf{tank}\theta \hat{k} \times)^2 = \mathbf{seck}^2\theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -\tan\theta & 0 \\ \tan\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} \sec\theta & 0 & 0 \\ 0 & \sec\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2$$

(3) Compound angle formula for sink

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

$$\mathbf{sink}(\alpha + \beta) \hat{k} \times = \mathbf{cosk}\alpha \mathbf{sink}\beta \hat{k} \times + \mathbf{cosk}\beta \mathbf{sink}\alpha \hat{k} \times$$

$$\begin{pmatrix} 0 & -\sin(\alpha + \beta) & 0 \\ \sin(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(4) Compound angle formula for cosk

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$\mathbf{cosk}(\alpha + \beta) = \mathbf{cosk}\alpha \mathbf{cosk}\beta + (\mathbf{sink}\alpha \hat{k} \times)(\mathbf{sink}\beta \hat{k} \times)$$

$$\begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(5) Compound angle formula for tank

$$\tan(\alpha + \beta) = (\tan\alpha + \tan\beta)(1 - \tan\alpha \tan\beta)^{-1}$$

$$\mathbf{tank}(\alpha + \beta) \hat{k} \times = (\mathbf{tank}\alpha \hat{k} \times + \mathbf{tank}\beta \hat{k} \times) (1 - (\mathbf{tank}\alpha \hat{k} \times)(\mathbf{tank}\beta \hat{k} \times))^{-1}$$

$$\begin{pmatrix} 0 & -\tan(\alpha + \beta) & 0 \\ \tan(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\tan\alpha - \tan\beta & 0 \\ \tan\alpha + \tan\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 - \tan\alpha \tan\beta & 0 & 0 \\ 0 & 1 - \tan\alpha \tan\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

(6) Double angle formula for sink

$$\sin(2\alpha) = 2\sin\alpha \cos\alpha$$

$$\mathbf{sink}(2\alpha) \hat{k} \times = 2\mathbf{cosk}\alpha \mathbf{sink}\alpha \hat{k} \times$$

$$\begin{pmatrix} 0 & -\sin 2\alpha & 0 \\ \sin 2\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(7) Double angle formula for **cosk**

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha \quad \mathbf{cosk}(2\alpha) = \mathbf{cosk}^2 \alpha + (\mathbf{sink} \alpha \hat{k} \times)^2$$

$$\begin{pmatrix} \cos 2\alpha & 0 & 0 \\ 0 & \cos 2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos \alpha & 0 & 0 \\ 0 & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 + \begin{pmatrix} 0 & -\sin \alpha & 0 \\ \sin \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2$$

(8) Double angle formula for **tank**

$$\tan(2\alpha) = 2(\tan \alpha)(1 - \tan^2 \alpha)^{-1} \quad \mathbf{tank}(2\alpha) \hat{k} \times = (2\mathbf{tank} \alpha \hat{k} \times)(1 - (\mathbf{tank} \alpha \hat{k} \times)^2)^{-1}$$

$$\begin{pmatrix} 0 & -\tan 2\alpha & 0 \\ \tan 2\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & -\tan \alpha & 0 \\ \tan \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 - \tan^2 \alpha & 0 & 0 \\ 0 & 1 - \tan^2 \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

(9) Products of **sink**

$$2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta) \quad 2(\mathbf{sink} \alpha \hat{k} \times)(\mathbf{sink} \beta \hat{k} \times) = \mathbf{cosk}(\alpha - \beta) - \mathbf{cosk}(\alpha + \beta)$$

$$2 \begin{pmatrix} 0 & -\sin \alpha & 0 \\ \sin \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin \beta & 0 \\ \sin \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\alpha - \beta) & 0 & 0 \\ 0 & \cos(\alpha - \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(10) Products of **cosk**

$$2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta) \quad 2\mathbf{cosk} \alpha \mathbf{cosk} \beta = \mathbf{cosk}(\alpha - \beta) + \mathbf{cosk}(\alpha + \beta)$$

$$2 \begin{pmatrix} \cos \alpha & 0 & 0 \\ 0 & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & 0 \\ 0 & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha - \beta) & 0 & 0 \\ 0 & \cos(\alpha - \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(11) Products of **sink** and **cosk**

$$2 \sin \alpha \cos \beta = \sin(\alpha - \beta) + \sin(\alpha + \beta) \quad 2(\mathbf{sink} \alpha \hat{k} \times) \mathbf{cosk} \beta = \mathbf{sink}(\alpha - \beta) \hat{k} \times + \mathbf{sink}(\alpha + \beta) \hat{k} \times$$

$$2 \begin{pmatrix} 0 & -\sin \alpha & 0 \\ \sin \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & 0 \\ 0 & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\sin(\alpha - \beta) & 0 \\ \sin(\alpha - \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sin(\alpha + \beta) & 0 \\ \sin(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Over the next few pages are the proofs for firstly

(1) $\mathbf{cosk}^2\theta - (\mathbf{sink}\theta \hat{k} \times)^2 = 1$ and then

(2) $1 - (\mathbf{tank} \theta \hat{k} \times)^2 = \mathbf{seck}^2\theta$ and then

(3) $\mathbf{sink}(\alpha + \beta) \hat{k} \times = \mathbf{cosk} \alpha \mathbf{sink} \beta \hat{k} \times + \mathbf{cosk} \beta \mathbf{sink} \alpha \hat{k} \times$ and finally

(4) $\mathbf{cosk}(\alpha + \beta) = \mathbf{cosk} \alpha \mathbf{cosk} \beta + (\mathbf{sink} \alpha \hat{k} \times)(\mathbf{sink} \beta \hat{k} \times)$

from which each of the others can be obtained: half angle formulae by putting $\beta = \alpha$, and adding and subtracting the compound angle formulae for products.

(1) Proof of $\text{cosk}^2\theta - (\text{sink}\theta \hat{k}\times)^2 = 1$, the basic circular relation for **sink** and **cosk**

$$\begin{aligned}\text{cosk}\theta &= \frac{1}{2}(\text{expk}(\theta) + \text{expk}(-\theta)) & \text{sink}\theta \hat{k}\times &= \frac{1}{2}(\text{expk}(\theta) - \text{expk}(-\theta)) \\ &= \frac{1}{2}(e^{\theta \hat{k}\times} + e^{-\theta \hat{k}\times}) & &= \frac{1}{2}(e^{\theta \hat{k}\times} - e^{-\theta \hat{k}\times})\end{aligned}$$

$$\begin{aligned}\text{cosk}^2\theta - (\text{sink}\theta \hat{k}\times)^2 &= \left(\frac{e^{\theta \hat{k}\times} + e^{-\theta \hat{k}\times}}{2}\right)^2 - \left(\frac{e^{\theta \hat{k}\times} - e^{-\theta \hat{k}\times}}{2}\right)^2 \\ &= \frac{e^{2\theta \hat{k}\times} + 2 + e^{-2\theta \hat{k}\times}}{4} - \frac{e^{2\theta \hat{k}\times} - 2 + e^{-2\theta \hat{k}\times}}{4} \\ &= \frac{(e^{2\theta \hat{k}\times} - e^{2\theta \hat{k}\times}) + (2 + 2) + (e^{-2\theta \hat{k}\times} - e^{-2\theta \hat{k}\times})}{4} \\ &= 1\end{aligned}$$

This is 4unit HSC and 1st and 2nd year complex algebra with $i = \hat{k}\times$. This is essentially complex algebra since it is for a single rotation axis (on the xy plane) so that $i = \hat{k}\times$ commutes with itself and its conjugate (and obviously, other scalars). Pretty much everything in complex algebra works just as well or better with the substitution $i = \hat{k}\times$. In the simplification of the squared terms, it is the **expk** and **expk'** that cancel so the basic circle relation holds irrespective of rotation axis which can be seen immediately without a formal change of base to another rotation axis. I change base here to j -hat and i -hat just to show that it all works out. All of these also work similarly for r -hat, θ -hat, and ϕ -hat in polar systems for example.

In terms of diadically expanded matrix operators

$$\begin{aligned}\begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 &= \begin{pmatrix} \cos^2\theta & 0 & 0 \\ 0 & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -\sin^2\theta & 0 & 0 \\ 0 & -\sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= I\end{aligned}$$

This is somewhat laborious to write out and I do it easily with my programmable calculator... or rather, I already know what the answers are because I have been at it for some years now [3 years for $i = \hat{k}\times$ and just short of 20 years in mathematical research (13 years since I submitted my more than adequate PhD that was never passed)], and also because it is so intuitive, and so just use my TI to check my working... mostly to check the directions of the rotations for changes of base. It is basic (elementary) advanced mathematics. I am doing it (fully developing the mathematics of **cosk**, **sink**, and **expk**, and those of i -hat and j -hat) because I want to do calculus with it. Now armed with the knowledge that $\hat{k}\times$ is the complex number (... not a number, operator), and the knowledge to change base to other principal axes (and confidence to rotate into polar bases), my next (or so) lot of work will be unimaginably good [I am expecting and hoping although I never know until I write it and work out all the details]. This one has turned out quite good and I am rather pleased with it, despite it being only elementary functions.

k -hat to j -hat principal rotation axis: **making a change of base** from the k -hat principal rotation axis to the j -hat principal rotation axis for the basic circular relation in **cosj** and **sinj** by applying the full rank complex number for the quarter rotation around i -hat **expi**($\pi/2$)

$$1 = \text{cosk}^2\theta - (\text{sink}\theta \hat{k} \times)^2$$

basic relation for $\hat{k} \times$ rotation

$$\text{expi}(\pi/2)(1)\text{expi}(-\pi/2) = \text{expi}(\pi/2) (\text{cosk}^2\theta - (\text{sink}\theta \hat{k} \times)^2) \text{expi}(-\pi/2)$$

change of base

$$\text{expi}(\pi/2)\text{expi}(-\pi/2) = \text{expi}(\pi/2) (\text{cosk}^2\theta) \text{expi}(-\pi/2) - \text{expi}(\pi/2) ((\text{sink}\theta \hat{k} \times)^2) \text{expi}(-\pi/2)$$

$$1 = \text{cosj}^2\theta - (\text{sinj}\theta \hat{j} \times)^2$$

basic relation for $\hat{j} \times$ rotation

that is, with change of base operator **expi** in diadically expanded matrix form $\text{expi}(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\text{expi}(\pi/2)(1)\text{expi}(-\pi/2) = \text{expi}(\pi/2) (\text{cosk}^2\theta - (\text{sink}\theta \hat{k} \times)^2) \text{expi}(-\pi/2)$$

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} I \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \left(\begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ I &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}^2 - \begin{pmatrix} 0 & 0 & \sin\theta \\ 0 & 0 & 0 \\ -\sin\theta & 0 & 0 \end{pmatrix}^2 \end{aligned}$$

which is the basic circular relation for the zx plane, $\text{cosj}^2\theta - (\text{sinj}\theta \hat{j} \times)^2 = 1$, after the change of base from the k -hat rotation axis to that of the j -hat rotation axis.

k -hat to i -hat principal rotation axis: **making a change of base** from the k -hat principal rotation axis to the i -hat principal rotation axis for the basic circular relation in **cosi** and **sini** by applying the full rank complex number for the antisense (transpose) quarter rotation around j -hat **expj** $(-\pi/2)$

$$1 = \text{cosk}^2\theta - (\text{sink}\theta \hat{k} \times)^2$$

basic relation for $\hat{k} \times$ rotation

$$\text{expj}(-\pi/2)(1)\text{expj}(\pi/2) = \text{expj}(-\pi/2) (\text{cosk}^2\theta - (\text{sink}\theta \hat{k} \times)^2) \text{expj}(\pi/2)$$

change of base

$$\text{expj}(-\pi/2)\text{expj}(\pi/2) = \text{expj}(-\pi/2) (\text{cosk}^2\theta) \text{expj}(\pi/2) - \text{expj}(-\pi/2) ((\text{sink}\theta \hat{k} \times)^2) \text{expj}(\pi/2)$$

$$1 = \text{cosi}^2\theta - (\text{sini}\theta \hat{i} \times)^2$$

basic relation for $\hat{i} \times$ rotation

that is, with change of base operator **expj** in diadically expanded matrix form $\text{expj}(-\pi/2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\text{expj}(-\pi/2)(1)\text{expj}(\pi/2) = \text{expj}(-\pi/2) (\text{cosk}^2\theta - (\text{sink}\theta \hat{k} \times)^2) \text{expj}(\pi/2)$$

$$\begin{aligned} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} I \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \left(\begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \right) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ I &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}^2 - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sin\theta \\ 0 & -\sin\theta & 0 \end{pmatrix}^2 \end{aligned}$$

which is the basic circular relation for the zx plane, $\text{cosi}^2\theta - (\text{sini}\theta \hat{i} \times)^2 = 1$, after the change of base from the k -hat rotation axis to that of the i -hat rotation axis.

Just to show that the circular relation $\text{cosk}^2\theta - (\text{sink}\theta \hat{k} \times)^2 = 1$ still holds good after the change of base to the j -hat and i -hat axes, doing the proofs for $\text{cosj}^2\theta - (\text{sinj}\theta \hat{j} \times)^2 = 1$ and $\text{cosi}^2\theta - (\text{sini}\theta \hat{i} \times)^2 = 1$ to show that these relations are equal to changing base to these principal axes from $\text{cosk}^2\theta - (\text{sink}\theta \hat{k} \times)^2 = 1$:

Proof of $\text{cosj}^2\theta - (\text{sinj}\theta \hat{j} \times)^2 = 1$, the basic circular relation for **sinj** and **cosj**

$$\begin{aligned} \text{cosj } \theta &= \frac{1}{2}(\text{expj}(\theta) + \text{expj}(-\theta)) & \text{sinj } \theta \hat{j} \times &= \frac{1}{2}(\text{expj}(\theta) - \text{expj}(-\theta)) \\ &= \frac{1}{2}(e^{\theta \hat{j} \times} + e^{-\theta \hat{j} \times}) & &= \frac{1}{2}(e^{\theta \hat{j} \times} - e^{-\theta \hat{j} \times}) \\ \text{cosj}^2\theta - (\text{sinj}\theta \hat{j} \times)^2 &= \left(\frac{e^{\theta \hat{j} \times} + e^{-\theta \hat{j} \times}}{2}\right)^2 - \left(\frac{e^{\theta \hat{j} \times} - e^{-\theta \hat{j} \times}}{2}\right)^2 = \left(\frac{e^{\theta \hat{j} \times} + e^{-\theta \hat{j} \times}}{2} + \frac{e^{\theta \hat{j} \times} - e^{-\theta \hat{j} \times}}{2}\right) \left(\frac{e^{\theta \hat{j} \times} + e^{-\theta \hat{j} \times}}{2} - \frac{e^{\theta \hat{j} \times} - e^{-\theta \hat{j} \times}}{2}\right) \\ &= e^{\theta \hat{j} \times} e^{-\theta \hat{j} \times} \\ &= 1 \end{aligned}$$

which is complex algebra for the zx plane (rather than xy plane for k -cross) since it is for a single rotation axis so that $\hat{j} \times$ commutes with itself and its conjugate (and other scalars) on this plane (the zx plane Argand diagram if there were such a thing) and is the same as that from a change of base from $\hat{k} \times$ to $\hat{j} \times$. In diadically expanded matrix basis operator form

$$\text{cosj}^2\theta - (\text{sinj}\theta \hat{j} \times)^2 = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}^2 - \begin{pmatrix} 0 & 0 & \sin\theta \\ 0 & 0 & 0 \\ -\sin\theta & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} \cos^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos^2\theta \end{pmatrix} - \begin{pmatrix} -\sin^2\theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proof of $\text{cosi}^2\theta - (\text{sini}\theta \hat{i} \times)^2 = 1$, the basic circular relation for **sini** and **cosi**

$$\begin{aligned} \text{cosi } \theta &= \frac{1}{2}(\text{expi}(\theta) + \text{expi}(-\theta)) & \text{sini } \theta \hat{i} \times &= \frac{1}{2}(\text{expi}(\theta) - \text{expi}(-\theta)) \\ &= \frac{1}{2}(e^{\theta \hat{i} \times} + e^{-\theta \hat{i} \times}) & &= \frac{1}{2}(e^{\theta \hat{i} \times} - e^{-\theta \hat{i} \times}) \\ \text{cosi}^2\theta - (\text{sini}\theta \hat{i} \times)^2 &= \left(\frac{e^{\theta \hat{i} \times} + e^{-\theta \hat{i} \times}}{2}\right)^2 - \left(\frac{e^{\theta \hat{i} \times} - e^{-\theta \hat{i} \times}}{2}\right)^2 = \left(\frac{e^{\theta \hat{i} \times} + e^{-\theta \hat{i} \times}}{2} + \frac{e^{\theta \hat{i} \times} - e^{-\theta \hat{i} \times}}{2}\right) \left(\frac{e^{\theta \hat{i} \times} + e^{-\theta \hat{i} \times}}{2} - \frac{e^{\theta \hat{i} \times} - e^{-\theta \hat{i} \times}}{2}\right) \\ &= e^{\theta \hat{i} \times} e^{-\theta \hat{i} \times} \\ &= 1 \end{aligned}$$

which is complex algebra for the yz plane (rather than xy plane for k -cross or the zx plane for j -cross) since it is for a single rotation axis so that $\hat{i} \times$ commutes with itself and its conjugate (and other scalars)

$$\text{cosi}^2\theta - (\text{sini}\theta \hat{i} \times)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}^2 - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta & 0 \\ 0 & 0 & \cos^2\theta \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin^2\theta & 0 \\ 0 & 0 & -\sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

yep, all works as it should.

(2) The mathematics for the basic circular relation in **tank**, **tanj**, and **tani** in terms of **seck**, **secj**, **seci** is similar and so is omitted. It is demonstrated with a programmable calculator by the same methods as above, or just divide through by **cosk**, **cosj**, or **cosi** or their squares (or equivalently by **seck**, **secj**, or **seci**) as applicable to the relevant rotation axis and rotational plane (since **cosk**, **cosj**, and **cosi** are full rank). Note that ‘divide through by’ here means multiply by the inverse since the functions are R^3 matrices.

Actually I'll do just one of them here... this is the maths for the basis tangent circular relation in **tank** and **seck** from that in **cosk** and **sink**: multiplying both sides of $\mathbf{cosk}^2\theta - (\mathbf{sink}\theta \hat{k} \times)^2 = 1$ by $\mathbf{seck}^2\theta$

$$\mathbf{seck}^2\theta(\mathbf{cosk}^2\theta - (\mathbf{sink}\theta \hat{k} \times)^2) = \mathbf{seck}^2\theta$$

$$1 - (\mathbf{tank}\theta \hat{k} \times)^2 = \mathbf{seck}^2\theta$$

where $\mathbf{seck}^2\theta \mathbf{cosk}^2\theta = 1$ and $\mathbf{seck}^2\theta (\mathbf{sink}\theta \hat{k} \times)^2 = (\mathbf{tank}\theta \hat{k} \times)^2$

$$\text{that is, from } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} \cos^2\theta & 0 & 0 \\ 0 & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -\sin^2\theta & 0 & 0 \\ 0 & -\sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{multiplying both sides by } \mathbf{cosk}^{-2}\theta = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-2} = \begin{pmatrix} \cos^{-2}\theta & 0 & 0 \\ 0 & \cos^{-2}\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sec^2\theta & 0 & 0 \\ 0 & \sec^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{seck}^2\theta$$

the $\mathbf{cosk}^2\theta - (\mathbf{sink}\theta \hat{k} \times)^2 = 1$ relation converts to $1 - (\mathbf{tank}\theta \hat{k} \times)^2 = \mathbf{seck}^2\theta$ by multiplication through by $\mathbf{seck}^2\theta$

$$\begin{aligned} & \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-2} \left(\begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 - \begin{pmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \right) \\ & \begin{pmatrix} \sec^2\theta & 0 & 0 \\ 0 & \sec^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sec^2\theta & 0 & 0 \\ 0 & \sec^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos^2\theta & 0 & 0 \\ 0 & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \sec^2\theta & 0 & 0 \\ 0 & \sec^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\sin^2\theta & 0 \\ \sin^2\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & \begin{pmatrix} \sec^2\theta & 0 & 0 \\ 0 & \sec^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -\tan^2\theta & 0 \\ \tan^2\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & \mathbf{seck}^2\theta = 1 - (\mathbf{tank}\theta \hat{k} \times)^2 \end{aligned}$$

which is the required relation.

(3) Proof of $\mathbf{sink}(\alpha + \beta)\hat{k} \times = \mathbf{cosk}\alpha \mathbf{sink}\beta \hat{k} \times + \mathbf{cosk}\beta \mathbf{sink}\alpha \hat{k} \times$, the compound angle formula for sine

$$\begin{aligned} \mathbf{cosk}\alpha \mathbf{sink}\beta \hat{k} \times + \mathbf{cosk}\beta \mathbf{sink}\alpha \hat{k} \times &= \frac{1}{2} \left(e^{\alpha \hat{k} \times} + e^{-\alpha \hat{k} \times} \right) \frac{1}{2} \left(e^{\beta \hat{k} \times} - e^{-\beta \hat{k} \times} \right) + \frac{1}{2} \left(e^{\beta \hat{k} \times} + e^{-\beta \hat{k} \times} \right) \frac{1}{2} \left(e^{\alpha \hat{k} \times} - e^{-\alpha \hat{k} \times} \right) \\ &= \frac{1}{4} \left(e^{\alpha \hat{k} \times} e^{\beta \hat{k} \times} - e^{\alpha \hat{k} \times} e^{-\beta \hat{k} \times} + e^{-\alpha \hat{k} \times} e^{\beta \hat{k} \times} - e^{-\alpha \hat{k} \times} e^{-\beta \hat{k} \times} \right) \\ &\quad + \frac{1}{4} \left(e^{\beta \hat{k} \times} e^{\alpha \hat{k} \times} - e^{\beta \hat{k} \times} e^{-\alpha \hat{k} \times} + e^{-\beta \hat{k} \times} e^{\alpha \hat{k} \times} - e^{-\beta \hat{k} \times} e^{-\alpha \hat{k} \times} \right) \\ &= \frac{1}{4} \left(e^{\alpha \hat{k} \times} e^{\beta \hat{k} \times} - e^{-\alpha \hat{k} \times} e^{-\beta \hat{k} \times} \right) \\ &= \frac{1}{4} \left(e^{(\alpha + \beta) \hat{k} \times} - e^{-(\alpha + \beta) \hat{k} \times} \right) \\ &= \mathbf{sink}(\alpha + \beta) \hat{k} \times \end{aligned}$$

where $e^{\alpha \hat{k} \times}$ and $e^{\beta \hat{k} \times}$ and their conjugates commute since they are basis exponentials for the same rotation axis and rotational operator $\hat{k} \times$. In diadically expanded matrix operators

$$\begin{aligned}
\text{cosk}\alpha \text{sink}\beta \hat{k} \times + \text{cosk}\beta \text{sink}\alpha \hat{k} \times &= \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\cos\alpha \sin\beta & 0 \\ \cos\alpha \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\cos\beta \sin\alpha & 0 \\ \cos\beta \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\sin(\alpha + \beta) & 0 \\ \sin(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \text{sink}(\alpha + \beta) \hat{k} \times
\end{aligned}$$

see, it all works like a dream. Quite pleased... looks nice too. Now changing base to the other principal rotation axes:

k -hat to j -hat principal rotation axis: changing base to the j -hat rotation axes with $\text{expi}(\pi/2)$ and $\text{expi}(-\pi/2)$

$$\begin{aligned}
\text{sink}(\alpha + \beta) \hat{k} \times &= \text{cosk}\alpha \text{sink}\beta \hat{k} \times + \text{cosk}\beta \text{sink}\alpha \hat{k} \times && \text{compound angle formula for sink} \\
\text{expi}(\pi/2) (\text{sink}(\alpha + \beta) \hat{k} \times) \text{expi}(-\pi/2) &= \text{expi}(\pi/2) (\text{cosk}\alpha \text{sink}\beta \hat{k} \times + \text{cosk}\beta \text{sink}\alpha \hat{k} \times) \text{expi}(-\pi/2) && \text{change of base} \\
\text{sinj}(\alpha + \beta) \hat{j} \times &= \text{expi}(\pi/2) (\text{cosk}\alpha \text{sink}\beta \hat{k} \times) \text{expi}(-\pi/2) + \text{expi}(\pi/2) (\text{cosk}\beta \text{sink}\alpha \hat{k} \times) \text{expi}(-\pi/2) \\
&= \text{expi}(\pi/2) (\text{cosk}\alpha) \text{expi}(-\pi/2) \text{expi}(\pi/2) (\text{sink}\beta \hat{k} \times) \text{expi}(-\pi/2) \\
&\quad + \text{expi}(\pi/2) (\text{cosk}\beta) \text{expi}(-\pi/2) \text{expi}(\pi/2) (\text{sink}\alpha \hat{k} \times) \text{expi}(-\pi/2) \\
\text{sinj}(\alpha + \beta) \hat{j} \times &= \text{cosj}\alpha \text{sinj}\beta \hat{j} \times + \text{cosj}\beta \text{sinj}\alpha \hat{j} \times && \text{compound angle formula for sinj}
\end{aligned}$$

In matrix operators

$$\begin{aligned}
\text{expi}(\pi/2) (\text{sink}(\alpha + \beta) \hat{k} \times) \text{expi}(-\pi/2) &= \text{expi}(\pi/2) (\text{cosk}\alpha \text{sink}\beta \hat{k} \times + \text{cosk}\beta \text{sink}\alpha \hat{k} \times) \text{expi}(-\pi/2) \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin(\alpha + \beta) & 0 \\ \sin(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & \sin(\alpha + \beta) \\ 0 & 0 & 0 \\ -\sin(\alpha + \beta) & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & \sin(\alpha + \beta) \\ 0 & 0 & 0 \\ -\sin(\alpha + \beta) & 0 & 0 \end{pmatrix} &= \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & \sin\beta \\ 0 & 0 & 0 \\ -\sin\beta & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 0 & 0 & \sin\alpha \\ 0 & 0 & 0 \\ -\sin\alpha & 0 & 0 \end{pmatrix} \\
\text{sinj}(\alpha + \beta) \hat{j} \times &= \text{cosj}\alpha \text{sinj}\beta \hat{j} \times + \text{cosj}\beta \text{sinj}\alpha \hat{j} \times
\end{aligned}$$

which is the right answer for the sine compound angle formula for the j -hat rotation axis base changed from the k -hat rotation axis.

k -hat to i -hat principal rotation axis: changing base to the i -hat principal rotation axes with $\text{expj}(-\pi/2)$ and $\text{expj}(\pi/2)$

$$\begin{aligned}
\text{sink}(\alpha + \beta) \hat{k} \times &= \text{cosk}\alpha \text{sink}\beta \hat{k} \times + \text{cosk}\beta \text{sink}\alpha \hat{k} \times && \text{compound angle formula for sink} \\
\text{expj}(-\pi/2) (\text{sink}(\alpha + \beta) \hat{k} \times) \text{expj}(\pi/2) &= \text{expj}(-\pi/2) (\text{cosk}\alpha \text{sink}\beta \hat{k} \times + \text{cosk}\beta \text{sink}\alpha \hat{k} \times) \text{expj}(\pi/2) && \text{change of base} \\
\text{sini}(\alpha + \beta) \hat{i} \times &= \text{expj}(-\pi/2) (\text{cosk}\alpha \text{sink}\beta \hat{k} \times) \text{expj}(\pi/2) + \text{expj}(-\pi/2) (\text{cosk}\beta \text{sink}\alpha \hat{k} \times) \text{expj}(\pi/2) \\
&= \text{expj}(-\pi/2) (\text{cosk}\alpha) \text{expj}(\pi/2) \text{expj}(-\pi/2) (\text{sink}\beta \hat{k} \times) \text{expj}(\pi/2) \\
&\quad + \text{expj}(-\pi/2) (\text{cosk}\beta) \text{expj}(\pi/2) \text{expj}(-\pi/2) (\text{sink}\alpha \hat{k} \times) \text{expj}(\pi/2) \\
\text{sini}(\alpha + \beta) \hat{i} \times &= \text{cosi}\alpha \text{sini}\beta \hat{i} \times + \text{cosi}\beta \text{sini}\alpha \hat{i} \times && \text{compound angle formula for sini}
\end{aligned}$$

In matrix operators

$$\begin{aligned}
&\text{expj}(-\pi/2) (\text{sink}(\alpha + \beta) \hat{k} \times) \text{expj}(\pi/2) = \text{expj}(-\pi/2) (\text{cosk}\alpha \text{sink}\beta \hat{k} \times + \text{cosk}\beta \text{sink}\alpha \hat{k} \times) \text{expj}(\pi/2) \\
&\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin(\alpha + \beta) & 0 \\ \sin(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\alpha + \beta) \\ 0 & \sin(\alpha + \beta) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\alpha + \beta) \\ 0 & \sin(\alpha + \beta) & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\beta \\ 0 & \sin\beta & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha \\ 0 & \sin\alpha & 0 \end{pmatrix} \\
&\text{sini}(\alpha + \beta) \hat{i} \times = \text{cosi}\alpha \text{sini}\beta \hat{i} \times + \text{cosi}\beta \text{sini}\alpha \hat{i} \times
\end{aligned}$$

which is the right answer for the sine compound angle formula for the i -hat rotation axis base changed from the k -hat rotation axis.

(4) Proof of $\text{cosk}(\alpha + \beta) = \text{cosk}\alpha \text{cosk}\beta + (\text{sink}\alpha \hat{k} \times)(\text{sink}\beta \hat{k} \times)$, the compound angle formula for cosine

$$\begin{aligned}
\text{cosk}\alpha \text{cosk}\beta + (\text{sink}\alpha \hat{k} \times)(\text{sink}\beta \hat{k} \times) &= \frac{1}{2} \left(e^{\alpha \hat{k} \times} + e^{-\alpha \hat{k} \times} \right) \frac{1}{2} \left(e^{\beta \hat{k} \times} + e^{-\beta \hat{k} \times} \right) + \frac{1}{2} \left(e^{\alpha \hat{k} \times} - e^{-\alpha \hat{k} \times} \right) \frac{1}{2} \left(e^{\beta \hat{k} \times} - e^{-\beta \hat{k} \times} \right) \\
&= \frac{1}{4} \left(e^{\alpha \hat{k} \times} e^{\beta \hat{k} \times} + e^{\alpha \hat{k} \times} e^{-\beta \hat{k} \times} + e^{-\alpha \hat{k} \times} e^{\beta \hat{k} \times} + e^{-\alpha \hat{k} \times} e^{-\beta \hat{k} \times} \right) \\
&\quad + \frac{1}{4} \left(e^{\alpha \hat{k} \times} e^{\beta \hat{k} \times} - e^{\alpha \hat{k} \times} e^{-\beta \hat{k} \times} - e^{-\alpha \hat{k} \times} e^{\beta \hat{k} \times} + e^{-\alpha \hat{k} \times} e^{-\beta \hat{k} \times} \right) \\
&= \frac{1}{2} \left(e^{\alpha \hat{k} \times} e^{\beta \hat{k} \times} + e^{-\alpha \hat{k} \times} e^{-\beta \hat{k} \times} \right) \\
&= \frac{1}{2} \left(e^{(\alpha + \beta) \hat{k} \times} + e^{-(\alpha + \beta) \hat{k} \times} \right) \\
&= \text{cosk}(\alpha + \beta)
\end{aligned}$$

In diadically expanded matrix operators

$$\begin{aligned}
\text{cosk}\alpha \text{cosk}\beta + (\text{sink}\alpha \hat{k} \times)(\text{sink}\beta \hat{k} \times) &= \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha \cos\beta & 0 & 0 \\ 0 & \cos\alpha \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -\sin\alpha \sin\beta & 0 & 0 \\ 0 & -\sin\alpha \sin\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \text{cosk}(\alpha + \beta)
\end{aligned}$$

Yep, again, it all works. Changing base to the other principal rotation axes:

k -hat to j -hat principal rotation axis: changing base to the j -hat rotation axes with $\text{expi}(\pi/2)$ and $\text{expi}(-\pi/2)$

$$\begin{aligned}
\text{cosk}(\alpha + \beta) &= \text{cosk}\alpha \text{cosk}\beta + (\text{sink}\alpha \hat{k} \times)(\text{sink}\beta \hat{k} \times) && \text{compound angle formula for cosk} \\
\text{expi}(\pi/2)(\text{cosk}(\alpha + \beta))\text{expi}(-\pi/2) &= \text{expi}(\pi/2)(\text{cosk}\alpha \text{cosk}\beta + (\text{sink}\alpha \hat{k} \times)(\text{sink}\beta \hat{k} \times))\text{expi}(-\pi/2) && \text{change of base} \\
\text{cosj}(\alpha + \beta) &= \text{expi}(\pi/2)(\text{cosk}\alpha \text{cosk}\beta)\text{expi}(-\pi/2) + \text{expi}(\pi/2)(\text{sink}\alpha \hat{k} \times)(\text{sink}\beta \hat{k} \times)\text{expi}(-\pi/2) \\
&= \text{expi}(\pi/2)(\text{cosk}\alpha \text{expi}(-\pi/2)\text{expi}(\pi/2)\text{cosk}\beta)\text{expi}(-\pi/2) \\
&\quad + \text{expi}(\pi/2)(\text{sink}\alpha \hat{k} \times)\text{expi}(-\pi/2)\text{expi}(\pi/2)(\text{sink}\beta \hat{k} \times)\text{expi}(-\pi/2) \\
&= \text{expi}(\pi/2)(\text{cosk}\alpha \text{expi}(-\pi/2)\text{expi}(\pi/2)\text{cosk}\beta)\text{expi}(-\pi/2) \\
&\quad + \text{expi}(\pi/2)(\text{sink}\alpha \hat{k} \times)\text{expi}(-\pi/2)\text{expi}(\pi/2)(\text{sink}\beta \hat{k} \times)\text{expi}(-\pi/2) \\
&= \text{cosj}\alpha \text{cosj}\beta + (\text{sinj}\alpha \hat{j} \times)(\text{sinj}\beta \hat{j} \times) && \text{compound angle formula for cosj}
\end{aligned}$$

In diadically expanded matrix form,

$$\begin{aligned}
\text{expi}(\pi/2)(\text{cosk}(\alpha + \beta))\text{expi}(-\pi/2) &= \text{expi}(\pi/2)(\text{cosk}\alpha \text{cosk}\beta + (\text{sink}\alpha \hat{k} \times)(\text{sink}\beta \hat{k} \times))\text{expi}(-\pi/2) \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
\begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos(\alpha + \beta) \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
\begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos(\alpha + \beta) \end{pmatrix} &= \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\beta \end{pmatrix} + \begin{pmatrix} 0 & 0 & \sin\alpha \\ 0 & 0 & 0 \\ -\sin\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \sin\beta \\ 0 & 0 & 0 \\ -\sin\beta & 0 & 0 \end{pmatrix} \\
\text{cosj}(\alpha + \beta) &= \text{cosj}\alpha \text{cosj}\beta + (\text{sinj}\alpha \hat{j} \times)(\text{sinj}\beta \hat{j} \times)
\end{aligned}$$

which is the right answer for the cosine compound angle formula for the j -hat rotation axis base changed from the k -hat rotation axis.

k -hat to i -hat principal rotation axis: changing base to the i -hat principal rotation axes with $\text{expi}(-\pi/2)$ and $\text{expi}(\pi/2)$

$$\begin{aligned}
\text{cosk}(\alpha + \beta) &= \text{cosk}\alpha \text{cosk}\beta + (\text{sink}\alpha \hat{k} \times)(\text{sink}\beta \hat{k} \times) && \text{compound angle formula for cosk} \\
\text{expj}(-\pi/2)(\text{cosk}(\alpha + \beta))\text{expj}(\pi/2) &= \text{expj}(-\pi/2)(\text{cosk}\alpha \text{cosk}\beta + (\text{sink}\alpha \hat{k} \times)(\text{sink}\beta \hat{k} \times))\text{expj}(\pi/2) && \text{change of base} \\
\text{cosi}(\alpha + \beta) &= \text{expj}(-\pi/2)(\text{cosk}\alpha \text{cosk}\beta)\text{expj}(\pi/2) + \text{expj}(-\pi/2)((\text{sink}\alpha \hat{k} \times)(\text{sink}\beta \hat{k} \times))\text{expj}(\pi/2) \\
&= \text{expj}(-\pi/2)(\text{cosk}\alpha \text{expj}(\pi/2)\text{expj}(-\pi/2)\text{cosk}\beta)\text{expj}(\pi/2) \\
&\quad + \text{expj}(-\pi/2)((\text{sink}\alpha \hat{k} \times)\text{expj}(\pi/2)\text{expj}(-\pi/2)(\text{sink}\beta \hat{k} \times))\text{expj}(\pi/2) \\
&= \text{expj}(-\pi/2)(\text{cosk}\alpha \text{expj}(\pi/2)\text{expj}(-\pi/2)\text{cosk}\beta)\text{expj}(\pi/2) \\
&\quad + \text{expj}(-\pi/2)((\text{sink}\alpha \hat{k} \times)\text{expj}(\pi/2)\text{expj}(-\pi/2)(\text{sink}\beta \hat{k} \times))\text{expj}(\pi/2) \\
&= \text{cosi}\alpha \text{cosi}\beta + (\text{sin}\alpha \hat{i} \times)(\text{sin}\beta \hat{i} \times) && \text{compound angle formula for cosi}
\end{aligned}$$

In diadically expanded matrix form,

$$\begin{aligned}
&\text{expj}(-\pi/2)(\text{cosk}(\alpha + \beta))\text{expj}(\pi/2) = \text{expj}(-\pi/2)(\text{cosk}\alpha \text{cosk}\beta + (\text{sink}\alpha \hat{k} \times)(\text{sink}\beta \hat{k} \times))\text{expj}(\pi/2) \\
&\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & \cos(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & \cos(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & \cos\beta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha \\ 0 & \sin\alpha & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\beta \\ 0 & \sin\beta & 0 \end{pmatrix} \\
&\text{cosi}(\alpha + \beta) = \text{cosi}\alpha \text{cosi}\beta + (\text{sin}\alpha \hat{i} \times)(\text{sin}\beta \hat{i} \times)
\end{aligned}$$

which is the right answer for the cosine compound angle formula for the i -hat rotation axis base changed from the k -hat rotation axis.

The relations for zx plane ($\hat{j} \times$ rotation axis) basis sines, cosines, and tangents (**sinj**, **cosj**, and **tanj**) are

(A) Circular Euler form for **cisj** = **expj**

$$\text{cosj}\theta + \text{sinj}\theta \hat{j} \times = \text{expj}\theta$$

(B) Circular Conjugate Euler form

$$\text{cosj}\theta - \text{sinj}\theta \hat{j} \times = \text{expj}(-\theta)$$

(1) Basic circular relation for **sinj** and **cosj**

$$\text{cosj}^2\theta - (\text{sinj}\theta \hat{j} \times)^2 = 1$$

(2) Basic circular relation for **tanj** and **secj**

$$1 - (\text{tanj}\theta \hat{j} \times)^2 = \text{secj}^2\theta$$

(3) Compound angle formula for **sinj**

$$\text{sinj}(\alpha + \beta) \hat{j} \times = \text{cosj}\alpha \text{sinj}\beta \hat{j} \times + \text{cosj}\beta \text{sinj}\alpha \hat{j} \times$$

(4) Compound angle formula for **cosj**

$$\text{cosj}(\alpha + \beta) = \text{cosj}\alpha \text{cosj}\beta + (\text{sinj}\alpha \hat{j} \times)(\text{sinj}\beta \hat{j} \times)$$

(5) Compound angle formula for **tanj**

$$\text{tanj}(\alpha + \beta) \hat{j} \times = (\text{tanj}\alpha \hat{j} \times + \text{tanj}\beta \hat{j} \times) (1 - (\text{tanj}\alpha \hat{j} \times)(\text{tanj}\beta \hat{j} \times))^{-1}$$

(6) Double angle formula for **sinj**

$$\text{sinj}(2\alpha) \hat{j} \times = 2\text{cosj}\alpha \text{sinj}\alpha \hat{j} \times$$

(7) Double angle formula for **cosj**

$$\text{cosj}(2\alpha) = \text{cosj}^2\alpha + (\text{sinj}\alpha \hat{j} \times)^2$$

(8) Double angle formula for **tanj**

$$\text{tanj}(2\alpha) \hat{j} \times = (2\text{tanj}\alpha \hat{j} \times) (1 - (\text{tanj}\alpha \hat{j} \times)^2)^{-1}$$

(9) Products of **sinj**

$$2(\text{sinj}\alpha \hat{j} \times)(\text{sinj}\beta \hat{j} \times) = \text{cosj}(\alpha - \beta) - \text{cosj}(\alpha + \beta)$$

(10) Products of **cosj**

$$2\text{cosj}\alpha \text{cosj}\beta = \text{cosj}(\alpha - \beta) + \text{cosj}(\alpha + \beta)$$

(11) Products of $\sin j$ and $\cos j$

$$2(\sin j \alpha \hat{j} \times) \cos j \beta = \sin j(\alpha - \beta) \hat{j} \times + \sin j(\alpha + \beta) \hat{j} \times$$

Diadically expanded into matrix base for the j -hat principal rotation axis:

nb, $\sin j$ is coupled with $j \times$

$$\sin j \theta = \begin{pmatrix} \sin \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sin \theta \end{pmatrix} \quad \sin j \theta \hat{j} \times = \begin{pmatrix} \sin \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sin \theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sin \theta \\ 0 & 0 & 0 \\ -\sin \theta & 0 & 0 \end{pmatrix}$$

(A) Circular Euler form for $\cos j = \exp j$

$$\cos \theta + i \sin \theta = e^{i\theta}$$

$$\cos \theta + \sin \theta \hat{k} \times = \exp \theta$$

$$\begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos \theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & \sin \theta \\ 0 & 0 & 0 \\ -\sin \theta & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$$

(B) Circular Conjugate Euler form

$$\cos \theta - i \sin \theta = e^{-i\theta}$$

$$\cos \theta - \sin \theta \hat{j} \times = \exp(-\theta)$$

$$\begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos \theta \end{pmatrix} - \begin{pmatrix} 0 & 0 & \sin \theta \\ 0 & 0 & 0 \\ -\sin \theta & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}$$

(1) Basic circular relation for $\sin j$ and $\cos j$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\cos^2 \theta - (\sin \theta \hat{j} \times)^2 = 1$$

$$\begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos \theta \end{pmatrix}^2 - \begin{pmatrix} 0 & 0 & \sin \theta \\ 0 & 0 & 0 \\ -\sin \theta & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Basic circular relation for $\tan j$ and $\sec j$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 - (\tan \theta \hat{j} \times)^2 = \sec^2 \theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \tan \theta \\ 0 & 0 & 0 \\ -\tan \theta & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} \sec \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sec \theta \end{pmatrix}^2$$

(3) Compound angle formula for $\sin j$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \alpha \cos \beta$$

$$\sin j(\alpha + \beta) \hat{j} \times = \cos j \alpha \sin j \beta \hat{j} \times + \cos j \beta \sin j \alpha \hat{j} \times$$

$$\begin{pmatrix} 0 & 0 & \sin(\alpha + \beta) \\ 0 & 0 & 0 \\ -\sin(\alpha + \beta) & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos \alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & \sin \beta \\ 0 & 0 & 0 \\ -\sin \beta & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} 0 & 0 & \sin \alpha \\ 0 & 0 & 0 \\ -\sin \alpha & 0 & 0 \end{pmatrix}$$

(4) Compound angle formula for cosj

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$\text{cosj}(\alpha + \beta) = \text{cosj}\alpha \text{cosj}\beta + (\text{sinj}\alpha \hat{j} \times)(\text{sinj}\beta \hat{j} \times)$$

$$\begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\beta \end{pmatrix} + \begin{pmatrix} 0 & 0 & \sin\alpha \\ 0 & 0 & 0 \\ -\sin\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \sin\beta \\ 0 & 0 & 0 \\ -\sin\beta & 0 & 0 \end{pmatrix}$$

(5) Compound angle formula for tanj

$$\tan(\alpha + \beta) = (\tan\alpha + \tan\beta)(1 - \tan\alpha \tan\beta)^{-1}$$

$$\text{tanj}(\alpha + \beta) \hat{j} \times = (\text{tanj}\alpha \hat{j} \times + \text{tanj}\beta \hat{j} \times)(1 - (\text{tanj}\alpha \hat{j} \times)(\text{tanj}\beta \hat{j} \times))^{-1}$$

$$\begin{pmatrix} 0 & 0 & \tan(\alpha + \beta) \\ 0 & 0 & 0 \\ -\tan(\alpha + \beta) & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \tan\alpha + \tan\beta \\ 0 & 0 & 0 \\ -\tan\alpha - \tan\beta & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 - \tan\alpha \tan\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - \tan\alpha \tan\beta \end{pmatrix}^{-1}$$

(6) Double angle formula for sinj

$$\sin(2\alpha) = 2\sin\alpha \cos\alpha$$

$$\text{sinj}(2\alpha) \hat{j} \times = 2\text{cosj}\alpha \text{sinj}\alpha \hat{j} \times$$

$$\begin{pmatrix} 0 & 0 & \sin 2\alpha \\ 0 & 0 & 0 \\ -\sin 2\alpha & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & \sin\alpha \\ 0 & 0 & 0 \\ -\sin\alpha & 0 & 0 \end{pmatrix}$$

(7) Double angle formula for cosj

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

$$\text{cosj}(2\alpha) = \text{cosj}^2\alpha + (\text{sinj}\alpha \hat{j} \times)^2$$

$$\begin{pmatrix} \cos 2\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos 2\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix}^2 + \begin{pmatrix} 0 & 0 & \sin\alpha \\ 0 & 0 & 0 \\ -\sin\alpha & 0 & 0 \end{pmatrix}^2$$

(8) Double angle formula for tanj

$$\tan(2\alpha) = 2(\tan\alpha)(1 - \tan^2\alpha)^{-1}$$

$$\text{tanj}(2\alpha) \hat{j} \times = (2\text{tanj}\alpha \hat{j} \times)(1 - (\text{tanj}\alpha \hat{j} \times)^2)^{-1}$$

$$\begin{pmatrix} 0 & 0 & \tan 2\alpha \\ 0 & 0 & 0 \\ -\tan 2\alpha & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & \tan\alpha \\ 0 & 0 & 0 \\ -\tan\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 - \tan^2\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - \tan^2\alpha \end{pmatrix}^{-1}$$

(9) Products of sinj

$$2\sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

$$2(\text{sinj}\alpha \hat{j} \times)(\text{sinj}\beta \hat{j} \times) = \text{cosj}(\alpha - \beta) - \text{cosj}(\alpha + \beta)$$

$$2 \begin{pmatrix} 0 & 0 & \sin\alpha \\ 0 & 0 & 0 \\ -\sin\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \sin\beta \\ 0 & 0 & 0 \\ -\sin\beta & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\alpha - \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos(\alpha - \beta) \end{pmatrix} - \begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos(\alpha + \beta) \end{pmatrix}$$

(10) Products of $\cos j$

$$2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$$

$$2 \cos j \alpha \cos j \beta = \cos j(\alpha - \beta) + \cos j(\alpha + \beta)$$

$$2 \begin{pmatrix} \cos \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos \beta \end{pmatrix} = \begin{pmatrix} \cos(\alpha - \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos(\alpha - \beta) \end{pmatrix} + \begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos(\alpha + \beta) \end{pmatrix}$$

(11) Products of $\sin j$ and $\cos j$

$$2 \sin \alpha \cos \beta = \sin(\alpha - \beta) + \sin(\alpha + \beta)$$

$$2(\sin j \alpha \hat{j} \times) \cos j \beta = \sin j(\alpha - \beta) \hat{j} \times + \sin j(\alpha + \beta) \hat{j} \times$$

$$2 \begin{pmatrix} 0 & 0 & \sin \alpha \\ 0 & 0 & 0 \\ -\sin \alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos \beta \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sin(\alpha - \beta) \\ 0 & 0 & 0 \\ -\sin(\alpha - \beta) & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \sin(\alpha + \beta) \\ 0 & 0 & 0 \\ -\sin(\alpha + \beta) & 0 & 0 \end{pmatrix}$$

The relations for yz plane ($\hat{i} \times$ rotation axis) basis sines, cosines, and tangents (**sini**, **cosi**, and **tani**) are

(A) Circular Euler form for **cisi** = **expi**

$$\cos i \theta + \sin i \theta \hat{i} \times = \exp i \theta$$

(B) Circular Conjugate Euler form

$$\cos i \theta - \sin i \theta \hat{i} \times = \exp j(-\theta)$$

(1) Basic circular relation for **sini** and **cosi**

$$\cos^2 \theta - (\sin i \theta \hat{i} \times)^2 = 1$$

(2) Basic circular relation for **tani** and **seci**

$$1 - (\tan i \theta \hat{i} \times)^2 = \sec^2 \theta$$

(3) Compound angle formula for **sini**

$$\sin i(\alpha + \beta) \hat{i} \times = \cos i \alpha \sin i \beta \hat{i} \times + \cos i \beta \sin i \alpha \hat{i} \times$$

(4) Compound angle formula for **cosi**

$$\cos i(\alpha + \beta) = \cos i \alpha \cos i \beta + (\sin i \alpha \hat{i} \times)(\sin i \beta \hat{i} \times)$$

(5) Compound angle formula for **tani**

$$\tan i(\alpha + \beta) \hat{i} \times = (\tan i \alpha \hat{i} \times + \tan i \beta \hat{i} \times) (1 - (\tan i \alpha \hat{i} \times)(\tan i \beta \hat{i} \times))^{-1}$$

(6) Double angle formula for **sini**

$$\sin i(2\alpha) \hat{i} \times = 2 \cos i \alpha \sin i \alpha \hat{i} \times$$

(7) Double angle formula for **cosi**

$$\cos i(2\alpha) = \cos^2 \alpha + (\sin i \alpha \hat{i} \times)^2$$

(8) Double angle formula for **tani**

$$\tan i(2\alpha) \hat{i} \times = (2 \tan i \alpha \hat{i} \times) (1 - (\tan i \alpha \hat{i} \times)^2)^{-1}$$

(9) Products of **sini**

$$2(\sin i \alpha \hat{i} \times)(\sin i \beta \hat{i} \times) = \cos i(\alpha - \beta) - \cos i(\alpha + \beta)$$

(10) Products of **cosi**

$$2 \cos i \alpha \cos i \beta = \cos i(\alpha - \beta) + \cos i(\alpha + \beta)$$

(11) Products of **sini** and **cosi**

$$2(\sin i \alpha \hat{i} \times) \cos i \beta = \sin i(\alpha - \beta) \hat{i} \times + \sin i(\alpha + \beta) \hat{i} \times$$

Diadically expanded into matrix base for the i -hat principal rotation axis:

nb, **sini** is coupled with $i \times$

$$\sin i \theta = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \sin \theta \end{pmatrix} \quad \sin i \theta \hat{i} \times = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \sin \theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin \theta \\ 0 & \sin \theta & 0 \end{pmatrix}$$

(A) Circular Euler form for **cisi** = **expi**

$$\cos \theta + i \sin \theta = e^{i\theta}$$

$$\cos i \theta + \sin i \theta \hat{i} \times = \exp i \theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & 0 \\ 0 & 0 & \cos \theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin \theta \\ 0 & \sin \theta & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

(B) Circular Conjugate Euler form

$$\cos\theta + i\sin\theta = e^{i\theta}$$

$$\text{cosi}\theta - \text{sini}\theta \hat{i} \times = \text{expi}\theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

(1) Basic circular relation for **sini** and **cosi**

$$\cos^2\theta + \sin^2\theta = 1$$

$$\text{cosi}^2\theta - (\text{sini}\theta \hat{i} \times)^2 = 1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}^2 - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Basic circular relation for **tani** and **seci**

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 - (\text{tani}\theta \hat{i} \times)^2 = \text{seci}^2\theta$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\tan\theta \\ 0 & \tan\theta & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sec\theta & 0 \\ 0 & 0 & \sec\theta \end{pmatrix}^2$$

(3) Compound angle formula for **sini**

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

$$\text{sini}(\alpha + \beta) \hat{i} \times = \text{cosi}\alpha \text{sini}\beta \hat{i} \times + \text{cosi}\beta \text{sini}\alpha \hat{i} \times$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\alpha + \beta) \\ 0 & \sin(\alpha + \beta) & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\beta \\ 0 & \sin\beta & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha \\ 0 & \sin\alpha & 0 \end{pmatrix}$$

(4) Compound angle formula for **cosi**

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$\text{cosi}(\alpha + \beta) = \text{cosi}\alpha \text{cosi}\beta + (\text{sini}\alpha \hat{i} \times)(\text{sini}\beta \hat{i} \times)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & \cos(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & \cos\beta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha \\ 0 & \sin\alpha & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\beta \\ 0 & \sin\beta & 0 \end{pmatrix}$$

(5) Compound angle formula for **tani**

$$\tan(\alpha + \beta) = (\tan\alpha + \tan\beta)(1 - \tan\alpha \tan\beta)^{-1}$$

$$\text{tani}(\alpha + \beta) \hat{i} \times = (\text{tani}\alpha \hat{i} \times + \text{tani}\beta \hat{i} \times)(1 - (\text{tani}\alpha \hat{i} \times)(\text{tani}\beta \hat{i} \times))^{-1}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\tan(\alpha + \beta) \\ 0 & \tan(\alpha + \beta) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\tan\alpha - \tan\beta \\ 0 & \tan\alpha + \tan\beta & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \tan\alpha \tan\beta & 0 \\ 0 & 0 & 1 - \tan\alpha \tan\beta \end{pmatrix}^{-1}$$

(6) Double angle formula for $\mathbf{sini}$

$$\sin(2\alpha) = 2\sin\alpha \cos\alpha$$

$$\mathbf{sini}(2\alpha)\hat{i} \times = 2\mathbf{cosi}\alpha \mathbf{sini}\alpha \hat{i} \times$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin 2\alpha \\ 0 & \sin 2\alpha & 0 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha \\ 0 & \sin\alpha & 0 \end{pmatrix}$$

(7) Double angle formula for $\mathbf{cosi}$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

$$\mathbf{cosi}(2\alpha) = \mathbf{cosi}^2\alpha + (\mathbf{sini}\alpha \hat{i} \times)^2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\alpha & 0 \\ 0 & 0 & \cos 2\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix}^2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha \\ 0 & \sin\alpha & 0 \end{pmatrix}^2$$

(8) Double angle formula for $\mathbf{tani}$

$$\tan(2\alpha) = 2(\tan\alpha)(1 - \tan^2\alpha)^{-1}$$

$$\mathbf{tani}(2\alpha)\hat{i} \times = (2\mathbf{tani}\alpha \hat{i} \times)(1 - (\mathbf{tani}\alpha \hat{i} \times)^2)^{-1}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\tan 2\alpha \\ 0 & \tan 2\alpha & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\tan\alpha \\ 0 & \tan\alpha & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \tan^2\alpha & 0 \\ 0 & 0 & 1 - \tan^2\alpha \end{pmatrix}^{-1}$$

(9) Products of $\mathbf{sini}$

$$2\sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

$$2(\mathbf{sini}\alpha \hat{i} \times)(\mathbf{sini}\beta \hat{i} \times) = \mathbf{cosi}(\alpha - \beta) - \mathbf{cosi}(\alpha + \beta)$$

$$2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha \\ 0 & \sin\alpha & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\beta \\ 0 & \sin\beta & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha - \beta) & 0 \\ 0 & 0 & \cos(\alpha - \beta) \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & \cos(\alpha + \beta) \end{pmatrix}$$

(10) Products of $\mathbf{cosi}$

$$2\cos\alpha \cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$$

$$2\mathbf{cosi}\alpha \mathbf{cosi}\beta = \mathbf{cosi}(\alpha - \beta) + \mathbf{cosi}(\alpha + \beta)$$

$$2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & \cos\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha - \beta) & 0 \\ 0 & 0 & \cos(\alpha - \beta) \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & \cos(\alpha + \beta) \end{pmatrix}$$

(11) Products of $\mathbf{sini}$ and $\mathbf{cosi}$

$$2\sin\alpha \cos\beta = \sin(\alpha - \beta) + \sin(\alpha + \beta)$$

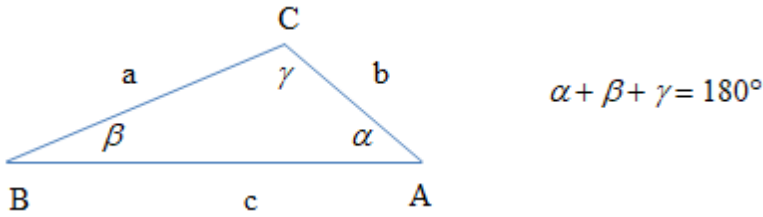
$$2(\mathbf{sini}\alpha \hat{i} \times)\mathbf{cosi}\beta = \mathbf{sini}(\alpha - \beta)\hat{i} \times + \mathbf{sini}(\alpha + \beta)\hat{i} \times$$

$$2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha \\ 0 & \sin\alpha & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & \cos\beta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\alpha - \beta) \\ 0 & \sin(\alpha - \beta) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\alpha + \beta) \\ 0 & \sin(\alpha + \beta) & 0 \end{pmatrix}$$

These also hold for the polar cylindrical and polar spherical bases for, obviously, both scleronomic [rotated] and rheonomic [rotating] bases. The difference here regarding whether a polar base is scleronomic or rheonomic is relevant for formulae when there is differential calculus [notably, time dots on the bases], otherwise the difference (scleronomic or rheonomic change of base) does not change the mathematics and does not need to be explicitly recognized, for example, for purely geometric or algebraic problems in the derived systems.

The Law of Basis Sines: The Law of **sink**, **sinj**, and **sini**

In how I have derived this, notably the intersection of circles, this immediately suggests to me that this would be of value to the GlobalPositioningSystem which relies on the intersection of spheres (circles of circles) to pinpoint a location. There are probably exact formulae for this already, the circle of the intersection of two spheres and the two solutions of the intersection with a third sphere determining the R^3 location, but the means here of obtaining the formulae (with **expk** and **expj** for instance) will make the calculations and resulting formulae more intuitive. It may also be possible to make these equations dynamic, since the constellation of satellites (and hence the interacting spheres) is in a constant state of orbital flux. It might also eventually be possible to describe it all in terms of differential equations for which the spheres and intersection of spheres are the solutions. This then means mathematical transfer functions which will probably give us many more tricks up our sleeve, mathematically, for those that maintain and advance the constellation. The orbital motions of the constellation is beautiful and similarly will be the differential equations.



Here I derive the Law of **sink** by equating basis exponentials as an intersection of circles [note that I haven't included the circles in the graphic although one might find it interesting to get their compass out [a proper one with a screw gear to keep the radii fixed] and draw the two circles noting the intersection at C... which is what I did whilst working it out on paper and I think provides for a good geometric intuition [of circles, radii, and frequency as well as the all important idea here of the intersection of circles of differing foci and radii]... I also used a compass to draw radii of convergence in my last complex calculus exam several years ago... I think I was the only one to do so in the exam room of several hundred candidates... compasses are a lost art form in mathematics, which yeah, is why I have mentioned it, here in the context of Archimedes belief that everything is circles... I also used a circular stencil set in my last rigid bodies exam which also worked out superbly (for torsional vibration)]. This is a generalization of the Law of sines to those of basis operators, **sink** in this instance, and hence converts the scalar algebraic law into that defined in the R^3 vector space. Change of base then converts the Law of **sink** to those of **sinj**, and **sini**.

Let B be the origin. Defining two circles, $C_1(\theta)$ centered at B with radius a and $C_2(\theta)$ centered at A with radius b . These two circles intersect twice. The intersection at C is the relation which produces the Law of **sink**.

The first circles is

$$C_1(\theta) = a \text{expk}(\theta) = a(\text{cosk}\theta + \text{sink}\theta \hat{k} \times) \text{ which intersects C at angle } \theta = \beta$$

$$C_1(\beta) = a \text{expk}(\beta) = a(\text{cosk}\beta + \text{sink}\beta \hat{k} \times)$$

The second circle is

$$C_2(\theta) = a \text{cosk}\beta + b \text{cosk}\alpha + b \text{expk}(\theta) = a \text{cosk}\beta + b \text{cosk}\alpha + b(\text{cosk}\theta + \text{sink}\theta \hat{k} \times) \text{ which intersects C at angle } \theta = \pi - \alpha$$

$$C_2(\pi - \alpha) = a \text{cosk}\beta + b \text{cosk}\alpha + b \text{expk}(\pi - \alpha) = a \text{cosk}\beta + b \text{cosk}\alpha + b(-\text{cosk}\alpha + \text{sink}\alpha \hat{k} \times) = a \text{cosk}\beta + b \text{sink}\alpha \hat{k} \times$$

Equating the two circles at point C, $C_1(\beta) = C_2(\pi - \alpha)$

$$C_1(\beta) = C_2(\pi - \alpha)$$

$$a \text{cosk}\beta + a \text{sink}\beta \hat{k} \times = a \text{cosk}\beta + b \text{sink}\alpha \hat{k} \times$$

$$a \text{sink}\beta \hat{k} \times = b \text{sink}\alpha \hat{k} \times$$

$$\frac{\text{sink}\beta \hat{k} \times}{b} = \frac{\text{sink}\alpha \hat{k} \times}{a}$$

Similarly drawing a circle with radius b centered at C and equating it for the intersection at A of that with radius c centered at B (this is trickier than the intersection at C but works out similarly)

$$\frac{\sin \gamma \hat{k} \times}{c} = \frac{\sin \beta \hat{k} \times}{b}$$

and since $(1/b)\sin \beta \hat{k} \times = (1/a)\sin \alpha \hat{k} \times$ and $(1/c)\sin \gamma \hat{k} \times = (1/b)\sin \beta \hat{k} \times$ by transitivity

$$\frac{\sin \alpha \hat{k} \times}{a} = \frac{\sin \gamma \hat{k} \times}{c}$$

so that

$$\frac{\sin \alpha \hat{k} \times}{a} = \frac{\sin \beta \hat{k} \times}{b} = \frac{\sin \gamma \hat{k} \times}{c}$$

which is the Law of **sink** for the xy plane. In terms of diadically expanded matrix operators

$$\frac{1}{a} \begin{pmatrix} 0 & -\sin \alpha & 0 \\ \sin \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{b} \begin{pmatrix} 0 & -\sin \beta & 0 \\ \sin \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{c} \begin{pmatrix} 0 & -\sin \gamma & 0 \\ \sin \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Making a change of base with **expi**($\pi/2$) and **expi**($-\pi/2$) for the Law of **sinj** for the zx plane

$$\frac{\sin \alpha \hat{k} \times}{a} = \frac{\sin \beta \hat{k} \times}{b} = \frac{\sin \gamma \hat{k} \times}{c} \quad \text{sink law}$$

$$\text{expi}(\pi/2) \frac{\sin \alpha \hat{k} \times}{a} \text{expi}(-\pi/2) = \text{expi}(\pi/2) \frac{\sin \beta \hat{k} \times}{b} \text{expi}(-\pi/2) = \text{expi}(\pi/2) \frac{\sin \gamma \hat{k} \times}{c} \text{expi}(-\pi/2) \quad \text{change of base}$$

$$\frac{\sin \alpha \hat{j} \times}{a} = \frac{\sin \beta \hat{j} \times}{b} = \frac{\sin \gamma \hat{j} \times}{c} \quad \text{sinj law}$$

$$\text{that is, with } k\text{-hat to } j\text{-hat principal rotation axis change of base matrix } \text{expi}(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} & \frac{1}{a} \begin{pmatrix} 0 & -\sin \alpha & 0 \\ \sin \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{b} \begin{pmatrix} 0 & -\sin \beta & 0 \\ \sin \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{c} \begin{pmatrix} 0 & -\sin \gamma & 0 \\ \sin \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{a} \begin{pmatrix} 0 & -\sin \alpha & 0 \\ \sin \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{b} \begin{pmatrix} 0 & -\sin \beta & 0 \\ \sin \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \dots \\ & \frac{1}{a} \begin{pmatrix} 0 & 0 & \sin \alpha \\ 0 & 0 & 0 \\ -\sin \alpha & 0 & 0 \end{pmatrix} = \frac{1}{b} \begin{pmatrix} 0 & 0 & \sin \beta \\ 0 & 0 & 0 \\ -\sin \beta & 0 & 0 \end{pmatrix} = \frac{1}{c} \begin{pmatrix} 0 & 0 & \sin \gamma \\ 0 & 0 & 0 \\ -\sin \gamma & 0 & 0 \end{pmatrix} \end{aligned}$$

(that with gamma in the second line of working omitted because of page width). Making a change of base with **expj**($-\pi/2$) and **expj**($\pi/2$) for the Law of **sini** for the yz plane

$$\frac{\sin \alpha \hat{k} \times}{a} = \frac{\sin \beta \hat{k} \times}{b} = \frac{\sin \gamma \hat{k} \times}{c} \quad \text{sink law}$$

$$\text{expj}(-\pi/2) \frac{\sin \alpha \hat{k} \times}{a} \text{expj}(\pi/2) = \text{expj}(-\pi/2) \frac{\sin \beta \hat{k} \times}{b} \text{expj}(\pi/2) = \text{expj}(-\pi/2) \frac{\sin \gamma \hat{k} \times}{c} \text{expj}(\pi/2) \quad \text{change of base}$$

$$\frac{\sin \alpha \hat{i} \times}{a} = \frac{\sin \beta \hat{i} \times}{b} = \frac{\sin \gamma \hat{i} \times}{c} \quad \text{sini law}$$

that is, with k -hat to i -hat principal rotation axis change of base matrix $\exp(-\pi/2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\begin{aligned} \frac{1}{a} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} &= \frac{1}{b} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{c} \begin{pmatrix} 0 & -\sin\gamma & 0 \\ \sin\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \frac{1}{a} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \frac{1}{b} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \dots \\ \frac{1}{a} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha \\ 0 & \sin\alpha & 0 \end{pmatrix} &= \frac{1}{b} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\beta \\ 0 & \sin\beta & 0 \end{pmatrix} = \frac{1}{c} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\gamma \\ 0 & \sin\gamma & 0 \end{pmatrix} \end{aligned}$$

(that with gamma in the second line of working similarly omitted because of page width). Note that the changes of base here are essentially group operations.

Summary for Law of basis sines for each principal rotation axis

$$\begin{aligned} k\text{-hat} \quad \frac{\sin\alpha \hat{k} \times}{a} &= \frac{\sin\beta \hat{k} \times}{b} = \frac{\sin\gamma \hat{k} \times}{c} & \frac{1}{a} \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} &= \frac{1}{b} \begin{pmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{c} \begin{pmatrix} 0 & -\sin\gamma & 0 \\ \sin\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ j\text{-hat} \quad \frac{\sin\alpha \hat{j} \times}{a} &= \frac{\sin\beta \hat{j} \times}{b} = \frac{\sin\gamma \hat{j} \times}{c} & \frac{1}{a} \begin{pmatrix} 0 & 0 & \sin\alpha \\ 0 & 0 & 0 \\ -\sin\alpha & 0 & 0 \end{pmatrix} &= \frac{1}{b} \begin{pmatrix} 0 & 0 & \sin\beta \\ 0 & 0 & 0 \\ -\sin\beta & 0 & 0 \end{pmatrix} = \frac{1}{c} \begin{pmatrix} 0 & 0 & \sin\gamma \\ 0 & 0 & 0 \\ -\sin\gamma & 0 & 0 \end{pmatrix} \\ i\text{-hat} \quad \frac{\sin\alpha \hat{i} \times}{a} &= \frac{\sin\beta \hat{i} \times}{b} = \frac{\sin\gamma \hat{i} \times}{c} & \frac{1}{a} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha \\ 0 & \sin\alpha & 0 \end{pmatrix} &= \frac{1}{b} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\beta \\ 0 & \sin\beta & 0 \end{pmatrix} = \frac{1}{c} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\gamma \\ 0 & \sin\gamma & 0 \end{pmatrix} \end{aligned}$$

The Law of Basis Tangents: The Law of **tank**, **tanj**, and **tani**

$$\begin{aligned} \frac{\tan(\alpha + \beta) \hat{k} \times}{a + b} &= \frac{\tan(\alpha - \beta) \hat{k} \times}{a - b} \\ \frac{1}{a + b} \begin{pmatrix} 0 & -\tan(\alpha + \beta) & 0 \\ \tan(\alpha + \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} &= \frac{1}{a - b} \begin{pmatrix} 0 & -\tan(\alpha - \beta) & 0 \\ \tan(\alpha - \beta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and with change of base from the k -hat principal rotation axis to that of the j -hat principal rotation axis

$$\begin{aligned} \frac{\tan(\alpha + \beta) \hat{j} \times}{a + b} &= \frac{\tan(\alpha - \beta) \hat{j} \times}{a - b} \\ \frac{1}{a + b} \begin{pmatrix} 0 & 0 & \tan(\alpha + \beta) \\ 0 & 0 & 0 \\ -\tan(\alpha + \beta) & 0 & 0 \end{pmatrix} &= \frac{1}{a - b} \begin{pmatrix} 0 & 0 & \tan(\alpha - \beta) \\ 0 & 0 & 0 \\ -\tan(\alpha - \beta) & 0 & 0 \end{pmatrix} \end{aligned}$$

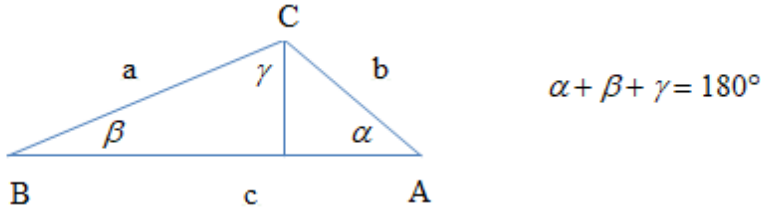
and with change of base from the k -hat principal rotation axis to that of the i -hat principal rotation axis

$$\frac{\tan(\alpha + \beta) \hat{i} \times}{a + b} = \frac{\tan(\alpha - \beta) \hat{i} \times}{a - b}$$

$$\frac{1}{a+b} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\tan(\alpha + \beta) \\ 0 & \tan(\alpha + \beta) & 0 \end{pmatrix} = \frac{1}{a-b} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\tan(\alpha - \beta) \\ 0 & \tan(\alpha - \beta) & 0 \end{pmatrix}$$

The Law of Basis Cosines: The Law of **cosk**, **cosj**, and **cosi**

The same method for the Law of **sink** has not worked for the Law of **cosk** because of issues with the rotational eigenvalue that result in a mathematical contradiction. So I have had to update the method by making the circle radii vectors. By doing so, the method works and gives a consistent outcome [and so updates, to an R^3 vector space, the Law of Cosines which was, as discussed in Clayton Dodge's book, first introduced to mathematics by Book II of Euclid's Elements].



Let h be the altitude at the orthogonal from vertex C and letting b_1 and b_2 be the lengths resulting from bisecting c with the orthogonal from C so that $c = b_1 + b_2$.

Noting that for this expression $c = a \text{cosk}\beta + b \text{cosk}\alpha$ doesn't work out.

$$(1) \quad c = \sqrt{b_1^2 - (h \hat{k} \times)^2} \text{cosk}\beta + \sqrt{b_2^2 - (h \hat{k} \times)^2} \text{cosk}\alpha \text{ instead results in a valid outcome}$$

From the **sink** law $b \text{sink}\alpha \hat{k} \times = a \text{sink}\beta \hat{k} \times$ which doesn't work either

$$\sqrt{b_1^2 - (h \hat{k} \times)^2} \text{sink}\beta \hat{k} \times = \sqrt{b_2^2 - (h \hat{k} \times)^2} \text{sink}\alpha \hat{k} \times \text{ gives the correct result.}$$

$$(2) \quad 0 = \sqrt{b_1^2 - (h \hat{k} \times)^2} \text{sink}\beta \hat{k} \times - \sqrt{b_2^2 - (h \hat{k} \times)^2} \text{sink}\alpha \hat{k} \times$$

Squaring (1) and (2) and subtracting

$$\begin{aligned} c^2 &= \left(\sqrt{b_1^2 - (h \hat{k} \times)^2} \text{cosk}\beta + \sqrt{b_2^2 - (h \hat{k} \times)^2} \text{cosk}\alpha \right)^2 + \left(\sqrt{b_1^2 - (h \hat{k} \times)^2} \text{sink}\beta \hat{k} \times - \sqrt{b_2^2 - (h \hat{k} \times)^2} \text{sink}\alpha \hat{k} \times \right)^2 \\ &= \left((b_1^2 - (h \hat{k} \times)^2) \text{cosk}^2\beta + (b_2^2 - (h \hat{k} \times)^2) \text{cosk}^2\alpha + 2\sqrt{b_1^2 - (h \hat{k} \times)^2} \sqrt{b_2^2 - (h \hat{k} \times)^2} \text{cosk}\beta \text{cosk}\alpha \right) \\ &\quad - \left((b_1^2 - (h \hat{k} \times)^2) (\text{sink}\beta \hat{k} \times)^2 + (b_2^2 - (h \hat{k} \times)^2) (\text{sink}\alpha \hat{k} \times)^2 - 2\sqrt{b_1^2 - (h \hat{k} \times)^2} \sqrt{b_2^2 - (h \hat{k} \times)^2} \text{sink}\alpha \hat{k} \times \text{sink}\beta \hat{k} \times \right) \\ &= (b_1^2 - (h \hat{k} \times)^2) (\text{cosk}^2\beta - \text{sink}^2\beta \hat{k} \times) + (b_2^2 - (h \hat{k} \times)^2) (\text{cosk}^2\alpha - \text{sink}^2\alpha \hat{k} \times) \\ &\quad + 2\sqrt{b_1^2 - (h \hat{k} \times)^2} \sqrt{b_2^2 - (h \hat{k} \times)^2} (\text{cosk}\beta \text{cosk}\alpha + \text{sink}\alpha \hat{k} \times \text{sink}\beta \hat{k} \times) \\ &= (b_1^2 - (h \hat{k} \times)^2) + (b_2^2 - (h \hat{k} \times)^2) + 2\sqrt{b_1^2 - (h \hat{k} \times)^2} \sqrt{b_2^2 - (h \hat{k} \times)^2} \text{cosk}(\alpha + \beta) \end{aligned}$$

which is the Law of **cosk** where $\alpha + \beta = \pi - \gamma$ and $\sqrt{b_1^2 - (h \hat{k} \times)^2}$ and $\sqrt{b_2^2 - (h \hat{k} \times)^2}$ are the magnitudes of the lengths BC and AC considered as vectors on the Argand $i = \hat{k} \times$ plane (real R^3 xy plane once directions are taken). There are three forms of operator in this operator equation

- $(b_1^2 - (h \hat{k} \times)^2)$ and $(b_2^2 - (h \hat{k} \times)^2)$ are double-crosses (repeated cross products $i^2 = -1$)
- **cosk**($\alpha + \beta$) is known
- $\sqrt{b_1^2 - (h \hat{k} \times)^2}$ and $\sqrt{b_2^2 - (h \hat{k} \times)^2}$ are yet to be discussed ... it all turns out good and works as it should

$$b_1^2 - (h \hat{k} \times)^2 = b_1^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - h^2 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = b_1^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - h^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b_1^2 + h^2 & 0 & 0 \\ 0 & b_1^2 + h^2 & 0 \\ 0 & 0 & b_1^2 \end{pmatrix} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & b_1^2 \end{pmatrix}$$

$$b_2^2 - (h \hat{k} \times)^2 = b_2^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - h^2 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = b_2^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - h^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b_2^2 + h^2 & 0 & 0 \\ 0 & b_2^2 + h^2 & 0 \\ 0 & 0 & b_2^2 \end{pmatrix} = \begin{pmatrix} b^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & b_2^2 \end{pmatrix}$$

$$\mathbf{cosk}(\alpha + \beta) = \begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For the first norm, $\sqrt{b_1^2 - (h \hat{k} \times)^2}$, expanding

$$\begin{aligned} \sqrt{b_1^2 - (h \hat{k} \times)^2} &= b_1 \sqrt{1 - (h/b_1)^2 (\hat{k} \times)^2} \\ &= b_1 \left(1 - \frac{1}{2} (h/b_1)^2 (\hat{k} \times)^2 - \frac{1}{8} (h/b_1)^4 (\hat{k} \times)^4 - \frac{1}{16} (h/b_1)^6 (\hat{k} \times)^6 - \frac{5}{128} (h/b_1)^8 (\hat{k} \times)^8 - \dots \right) \end{aligned}$$

Taking components

$$\begin{aligned} i\text{-hat component } \sqrt{b_1^2 - (h \hat{k} \times)^2} \hat{i} &= b_1 \left(1 - \frac{1}{2} (h/b_1)^2 (\hat{k} \times)^2 - \frac{1}{8} (h/b_1)^4 (\hat{k} \times)^4 - \frac{1}{16} (h/b_1)^6 (\hat{k} \times)^6 - \frac{5}{128} (h/b_1)^8 (\hat{k} \times)^8 - \dots \right) \hat{i} \\ &= \hat{i} b_1 \left(1 + \frac{1}{2} (h/b_1)^2 - \frac{1}{8} (h/b_1)^4 + \frac{1}{16} (h/b_1)^6 - \frac{5}{128} (h/b_1)^8 + \dots \right) \\ &= \hat{i} \sqrt{b_1^2 + h^2} \end{aligned}$$

$$\begin{aligned} j\text{-hat component } \sqrt{b_1^2 - (h \hat{k} \times)^2} \hat{j} &= b_1 \left(1 - \frac{1}{2} (h/b_1)^2 (\hat{k} \times)^2 - \frac{1}{8} (h/b_1)^4 (\hat{k} \times)^4 - \frac{1}{16} (h/b_1)^6 (\hat{k} \times)^6 - \frac{5}{128} (h/b_1)^8 (\hat{k} \times)^8 - \dots \right) \hat{j} \\ &= \hat{j} b_1 \left(1 + \frac{1}{2} (h/b_1)^2 - \frac{1}{8} (h/b_1)^4 + \frac{1}{16} (h/b_1)^6 - \frac{5}{128} (h/b_1)^8 + \dots \right) \\ &= \hat{j} \sqrt{b_1^2 + h^2} \end{aligned}$$

$$\begin{aligned} k\text{-hat component } \sqrt{b_1^2 - (h \hat{k} \times)^2} \hat{k} &= b_1 \left(1 - \frac{1}{2} (h/b_1)^2 (\hat{k} \times)^2 - \frac{1}{8} (h/b_1)^4 (\hat{k} \times)^4 - \frac{1}{16} (h/b_1)^6 (\hat{k} \times)^6 - \frac{5}{128} (h/b_1)^8 (\hat{k} \times)^8 - \dots \right) \hat{k} \\ &= \hat{k} b_1 (1 + 0 - 0 + 0 - 0 + \dots) \\ &= \hat{k} b_1 \end{aligned}$$

diadic expansion (vector components become columns)

$$\sqrt{b_1^2 - (h \hat{k} \times)^2} = \begin{pmatrix} \sqrt{b_1^2 + h^2} & 0 & 0 \\ 0 & \sqrt{b_1^2 + h^2} & 0 \\ 0 & 0 & b_1 \end{pmatrix} = \begin{pmatrix} \sqrt{a^2} & 0 & 0 \\ 0 & \sqrt{a^2} & 0 \\ 0 & 0 & b_1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b_1 \end{pmatrix}$$

where $b_1^2 + h^2 = a^2$. Similar maths for the second norm results in

$$\sqrt{b_2^2 - (h \hat{k} \times)^2} = \begin{pmatrix} \sqrt{b_2^2 + h^2} & 0 & 0 \\ 0 & \sqrt{b_2^2 + h^2} & 0 \\ 0 & 0 & b_2 \end{pmatrix} = \begin{pmatrix} \sqrt{b^2} & 0 & 0 \\ 0 & \sqrt{b^2} & 0 \\ 0 & 0 & b_2 \end{pmatrix} = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b_2 \end{pmatrix}$$

where $b_2^2 + h^2 = b^2$. So

$$c^2 = (b_1^2 - (h \hat{k} \times)^2) + (b_2^2 - (h \hat{k} \times)^2) + 2\sqrt{b_1^2 - (h \hat{k} \times)^2} \sqrt{b_2^2 - (h \hat{k} \times)^2} \mathbf{cosk}(\alpha + \beta)$$

$$\begin{aligned}
&= \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & b_1^2 \end{pmatrix} + \begin{pmatrix} b^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & b_2^2 \end{pmatrix} + 2 \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b_1 \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b_2 \end{pmatrix} \begin{pmatrix} \cos(\alpha + \beta) & 0 & 0 \\ 0 & \cos(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a^2 + b^2 + 2ab\cos(\alpha + \beta) & 0 & 0 \\ 0 & a^2 + b^2 + 2ab\cos(\alpha + \beta) & 0 \\ 0 & 0 & b_1^2 + b_2^2 + 2b_1b_2 \end{pmatrix} \\
&= \begin{pmatrix} a^2 + b^2 + 2ab\cos(\alpha + \beta) & 0 & 0 \\ 0 & a^2 + b^2 + 2ab\cos(\alpha + \beta) & 0 \\ 0 & 0 & c^2 \end{pmatrix}
\end{aligned}$$

where $b_1^2 + b_2^2 + 2b_1b_2 = (b_1 + b_2)^2 = c^2$. Therefore,

$$\begin{pmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} = \begin{pmatrix} a^2 + b^2 + 2ab\cos(\alpha + \beta) & 0 & 0 \\ 0 & a^2 + b^2 + 2ab\cos(\alpha + \beta) & 0 \\ 0 & 0 & c^2 \end{pmatrix}$$

which is the Law of **cosk**. The argument of the scalar cosine can now be changed to γ with $\alpha + \beta = \pi - \gamma$

$$\begin{pmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} = \begin{pmatrix} a^2 + b^2 - 2ab\cos\gamma & 0 & 0 \\ 0 & a^2 + b^2 - 2ab\cos\gamma & 0 \\ 0 & 0 & c^2 \end{pmatrix}$$

Changing base to the Laws of **cosj** and **cosi**:

For the Law of **cosj**, changing base from **cosk** with **expj**($\pi/2$)

$$\begin{aligned}
&\mathbf{expj}(\pi/2) \begin{pmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} \mathbf{expj}(-\pi/2) = \mathbf{expj}(\pi/2) \begin{pmatrix} a^2 + b^2 - 2ab\cos\gamma & 0 & 0 \\ 0 & a^2 + b^2 - 2ab\cos\gamma & 0 \\ 0 & 0 & c^2 \end{pmatrix} \mathbf{expj}(-\pi/2) \\
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a^2 + b^2 - 2ab\cos\gamma & 0 & 0 \\ 0 & a^2 + b^2 - 2ab\cos\gamma & 0 \\ 0 & 0 & c^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\
&\begin{pmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} = \begin{pmatrix} a^2 + b^2 - 2ab\cos\gamma & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & a^2 + b^2 - 2ab\cos\gamma \end{pmatrix}
\end{aligned}$$

For the Law of **cosi**, changing base from **cosk** with **expj**($-\pi/2$)

$$\begin{aligned}
&\mathbf{expj}(-\pi/2) \begin{pmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} \mathbf{expj}(\pi/2) = \mathbf{expj}(-\pi/2) \begin{pmatrix} a^2 + b^2 - 2ab\cos\gamma & 0 & 0 \\ 0 & a^2 + b^2 - 2ab\cos\gamma & 0 \\ 0 & 0 & c^2 \end{pmatrix} \mathbf{expj}(\pi/2) \\
&\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a^2 + b^2 - 2ab\cos\gamma & 0 & 0 \\ 0 & a^2 + b^2 - 2ab\cos\gamma & 0 \\ 0 & 0 & c^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&\begin{pmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} = \begin{pmatrix} c^2 & 0 & 0 \\ 0 & a^2 + b^2 - 2ab\cos\gamma & 0 \\ 0 & 0 & a^2 + b^2 - 2ab\cos\gamma \end{pmatrix}
\end{aligned}$$

Reflection and Transmission Coefficients: Unpolarized

Γ is the reflection coefficient and τ is the transmission coefficient

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \tau = \frac{2\eta_2}{\eta_2 + \eta_1} \quad \text{where the } \eta \text{ are the intrinsic impedances of the two media}$$

The reflection and transmission coefficients, worked out firstly by Fresnel, can be derived by assuming complex exponential electromagnetic fields. Now that it is known that complex exponentials are basis exponentials, **expk**, **expj**, and **expi**, I can say confidently that these will accurately describe electromagnetic fields in a more intuitive and complete way because these provide a means to work competently and confidently in R^3 . So before doing so, here I just state the formulae for the reflection and transmission coefficients which are defined in terms of cosines for polarized fields, and make the leap of faith by updating the scalar cosines [from Edminister and UNSW electromagnetics course notes] with **cosk** (and scalar secants with **seck**) given that every trigonometric identity has worked out by doing so with **sink**, **cosk**, and **tank** (and change of base to the other principal rotation axes). This is a sensible thing to do under the assumptions that the fields are functions of **expk**, **expj**, and **expi**.

Because **cosk** is full rank, all of the polarization formulae for the reflection and transmission coefficients work out fine. These would hold similarly well in the polar cylindrical and polar spherical coordinate systems. Note also the interesting linkage of reflection (and hence reflection coefficients) with $(\hat{k} \times)^2 = -1$, $(\hat{j} \times)^2 = -1$, and $(\hat{i} \times)^2 = -1$.

θ_t is the angle of transmission and θ_i is the angle of incidence.

Reflection and Transmission Coefficients: Perpendicular Polarization for the principal axes

$$\begin{aligned} \mathbf{\Gamma k}_{\perp} &= \frac{\eta_2 \mathbf{seck}\theta_t - \eta_1 \mathbf{seck}\theta_i}{\eta_2 \mathbf{seck}\theta_t + \eta_1 \mathbf{seck}\theta_i} & \mathbf{\tau k}_{\perp} &= \frac{2\eta_2 \mathbf{seck}\theta_t}{\eta_2 \mathbf{seck}\theta_t + \eta_1 \mathbf{seck}\theta_i} \\ &= \begin{pmatrix} \frac{\eta_2 \sec\theta_t - \eta_1 \sec\theta_i}{\eta_2 \sec\theta_t + \eta_1 \sec\theta_i} & 0 & 0 \\ 0 & \frac{\eta_2 \sec\theta_t - \eta_1 \sec\theta_i}{\eta_2 \sec\theta_t + \eta_1 \sec\theta_i} & 0 \\ 0 & 0 & \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \end{pmatrix} & & = \begin{pmatrix} \frac{2\eta_2 \sec\theta_t}{\eta_2 \sec\theta_t + \eta_1 \sec\theta_i} & 0 & 0 \\ 0 & \frac{2\eta_2 \sec\theta_t}{\eta_2 \sec\theta_t + \eta_1 \sec\theta_i} & 0 \\ 0 & 0 & \frac{2\eta_2}{\eta_2 + \eta_1} \end{pmatrix} \end{aligned}$$

Just to show that it works in R^3 and is non-singular for example, the maths of $\mathbf{\Gamma k}_{\perp}$ is

$$\begin{aligned} \mathbf{\Gamma k}_{\perp} &= \frac{\eta_2 \mathbf{seck}\theta_t - \eta_1 \mathbf{seck}\theta_i}{\eta_2 \mathbf{seck}\theta_t + \eta_1 \mathbf{seck}\theta_i} = (\eta_2 \mathbf{seck}\theta_t - \eta_1 \mathbf{seck}\theta_i)(\eta_2 \mathbf{seck}\theta_t + \eta_1 \mathbf{seck}\theta_i)^{-1} \\ &= \left(\eta_2 \begin{pmatrix} \sec\theta_t & 0 & 0 \\ 0 & \sec\theta_t & 0 \\ 0 & 0 & 1 \end{pmatrix} - \eta_1 \begin{pmatrix} \sec\theta_i & 0 & 0 \\ 0 & \sec\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \left(\eta_2 \begin{pmatrix} \sec\theta_t & 0 & 0 \\ 0 & \sec\theta_t & 0 \\ 0 & 0 & 1 \end{pmatrix} + \eta_1 \begin{pmatrix} \sec\theta_i & 0 & 0 \\ 0 & \sec\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1} \\ &= \begin{pmatrix} \eta_2 \sec\theta_t - \eta_1 \sec\theta_i & 0 & 0 \\ 0 & \eta_2 \sec\theta_t - \eta_1 \sec\theta_i & 0 \\ 0 & 0 & \eta_2 - \eta_1 \end{pmatrix} \begin{pmatrix} \eta_2 \sec\theta_t + \eta_1 \sec\theta_i & 0 & 0 \\ 0 & \eta_2 \sec\theta_t + \eta_1 \sec\theta_i & 0 \\ 0 & 0 & \eta_2 + \eta_1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \eta_2 \sec\theta_t - \eta_1 \sec\theta_i & 0 & 0 \\ 0 & \eta_2 \sec\theta_t - \eta_1 \sec\theta_i & 0 \\ 0 & 0 & \eta_2 - \eta_1 \end{pmatrix} \begin{pmatrix} \frac{1}{\eta_2 \sec\theta_t + \eta_1 \sec\theta_i} & 0 & 0 \\ 0 & \eta_2 \sec\theta_t + \eta_1 \sec\theta_i & 0 \\ 0 & 0 & \frac{1}{\eta_2 + \eta_1} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \frac{\eta_2 \sec\theta_t - \eta_1 \sec\theta_i}{\eta_2 \sec\theta_t + \eta_1 \sec\theta_i} & 0 & 0 \\ 0 & \frac{\eta_2 \sec\theta_t - \eta_1 \sec\theta_i}{\eta_2 \sec\theta_t + \eta_1 \sec\theta_i} & 0 \\ 0 & 0 & \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \end{pmatrix}$$

and where the maths of $\mathbf{rk}_\perp$ is similar. Change of base for the transverse $\perp$ j -hat coefficients for the $\hat{j} \times$ principal rotation axis

$$\mathbf{\Gamma j}_\perp = \mathbf{expi}(\pi/2) \mathbf{\Gamma k}_\perp \mathbf{expi}(-\pi/2) \quad \mathbf{rj}_\perp = \mathbf{expi}(\pi/2) \mathbf{rk}_\perp \mathbf{expi}(-\pi/2)$$

Change of base for the transverse $\perp$ i -hat coefficients for the $\hat{i} \times$ principal rotation axis

$$\mathbf{\Gamma i}_\perp = \mathbf{expj}(-\pi/2) \mathbf{\Gamma k}_\perp \mathbf{expj}(\pi/2) \quad \mathbf{ri}_\perp = \mathbf{expj}(-\pi/2) \mathbf{rk}_\perp \mathbf{expj}(\pi/2)$$

Reflection and Transmission Coefficients: Parallel Polarization for the principal axes

$$\mathbf{\Gamma k}_\parallel = \frac{\eta_2 \text{cosk}\theta_t - \eta_1 \text{cosk}\theta_i}{\eta_2 \text{cosk}\theta_t + \eta_1 \text{cosk}\theta_i} \quad \mathbf{rk}_\parallel = \frac{2\eta_2 \text{cosk}\theta_t}{\eta_2 \text{cosk}\theta_t + \eta_1 \text{cosk}\theta_i}$$

$$= \begin{pmatrix} \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} & 0 & 0 \\ 0 & \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} & 0 \\ 0 & 0 & \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \end{pmatrix} \quad = \begin{pmatrix} \frac{2\eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} & 0 & 0 \\ 0 & \frac{2\eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} & 0 \\ 0 & 0 & \frac{2\eta_2}{\eta_2 + \eta_1} \end{pmatrix}$$

where the maths of $\mathbf{\Gamma k}_\parallel$ is the same as $\mathbf{\Gamma k}_\perp$ with **seck** replaced with **cosk** (and so similarly with the scalars, here sec replaced with cos)

$$\mathbf{\Gamma k}_\parallel = \frac{\eta_2 \text{cosk}\theta_t - \eta_1 \text{cosk}\theta_i}{\eta_2 \text{cosk}\theta_t + \eta_1 \text{cosk}\theta_i} = (\eta_2 \text{cosk}\theta_t - \eta_1 \text{cosk}\theta_i)(\eta_2 \text{cosk}\theta_t + \eta_1 \text{cosk}\theta_i)^{-1}$$

$$= \left(\eta_2 \begin{pmatrix} \cos\theta_t & 0 & 0 \\ 0 & \cos\theta_t & 0 \\ 0 & 0 & 1 \end{pmatrix} - \eta_1 \begin{pmatrix} \cos\theta_i & 0 & 0 \\ 0 & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \left(\eta_2 \begin{pmatrix} \cos\theta_t & 0 & 0 \\ 0 & \cos\theta_t & 0 \\ 0 & 0 & 1 \end{pmatrix} + \eta_1 \begin{pmatrix} \cos\theta_i & 0 & 0 \\ 0 & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} \eta_2 \cos\theta_t - \eta_1 \cos\theta_i & 0 & 0 \\ 0 & \eta_2 \cos\theta_t - \eta_1 \cos\theta_i & 0 \\ 0 & 0 & \eta_2 - \eta_1 \end{pmatrix} \begin{pmatrix} \eta_2 \cos\theta_t + \eta_1 \cos\theta_i & 0 & 0 \\ 0 & \eta_2 \cos\theta_t + \eta_1 \cos\theta_i & 0 \\ 0 & 0 & \eta_2 + \eta_1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \eta_2 \cos\theta_t - \eta_1 \cos\theta_i & 0 & 0 \\ 0 & \eta_2 \cos\theta_t - \eta_1 \cos\theta_i & 0 \\ 0 & 0 & \eta_2 - \eta_1 \end{pmatrix} \begin{pmatrix} \frac{1}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} & 0 & 0 \\ 0 & \eta_2 \cos\theta_t + \eta_1 \cos\theta_i & 0 \\ 0 & 0 & \frac{1}{\eta_2 + \eta_1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} & 0 & 0 \\ 0 & \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} & 0 \\ 0 & 0 & \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \end{pmatrix}$$

Change of base for the parallel $\parallel$ j -hat coefficients for the $\hat{j} \times$ principal rotation axis

$$\mathbf{\Gamma j}_\parallel = \mathbf{expi}(\pi/2) \mathbf{\Gamma k}_\parallel \mathbf{expi}(-\pi/2) \quad \mathbf{rj}_\parallel = \mathbf{expi}(\pi/2) \mathbf{rk}_\parallel \mathbf{expi}(-\pi/2)$$

Change of base for the parallel $||$ i -hat coefficients for the $\hat{i} \times$ principal rotation axis

$$\mathbf{\Gamma}_{||} = \exp \mathbf{j}(-\pi/2) \mathbf{\Gamma}_{k||} \exp \mathbf{j}(\pi/2)$$

$$\mathbf{\tau}_{||} = \exp \mathbf{j}(-\pi/2) \mathbf{\tau}_{k||} \exp \mathbf{j}(\pi/2)$$

What might actually be better is to state the coefficients within polar sphericals and define them for the radius direction for arbitrarily azimuth and colatitude. Time will tell. This is one of my upcoming projects, when I start doing calculus with these functions of $i = \hat{k} \times$ for radar and the like... note here for interest the very special reflection $(\hat{r} \times)^2 = -1$ for a radar echo return (with a change of base for other directions of scatter). I am hoping to lay the groundwork for this here by making sure all of the elementary functions work mathematically as expected (and adjust these functions to the physics at a later stage when I start doing the calculus... I have been saying this regarding calculus for quite some time although have been unable to do so because it is harder [than the elementary mathematics] and I have been getting hardcore harassed by gang rape scum which has made hard resolving impossible as of these past few years... I was also hoping for a bursary or something [and peace and quiet] before giving away all of my tricks to the creative commons [given that it takes much effort]).

As a similar leap of faith in Snell's law by updating the sines with **sink** results in $n_i \text{sink} \theta_i \hat{k} \times = n_r \text{sink} \theta_r \hat{k} \times$. I think it is reasonable to assume that there will be a correspondence to the actual physics because of the electromagnetic link with **expk**, **expj**, and **expi** (which is an updated version of Archimedes conjecture that everything is circles). It looks just like the Law of **sink** with the angle of incidence θ_i and angle of refraction θ_r taking two of the triangle angles and the inverted indices of refraction (dielectric light velocity per unit of free-space light velocity v_i/c and v_r/c) being the triangle lengths (circle radii). So with

$$n_i = \frac{1}{v_i/c} \text{ and } n_r = \frac{1}{v_r/c}$$

(where light velocities, indices of refraction, and intrinsic impedances are all related by relative permittivities and permeabilities as per Maxwell theory)

it becomes

$$\frac{\text{sink} \theta_i \hat{k} \times}{v_i/c} = \frac{\text{sink} \theta_r \hat{k} \times}{v_r/c}$$

and in diadically expanded matrix operator form

$$\frac{1}{v_i/c} \begin{pmatrix} 0 & -\sin \theta_i & 0 \\ \sin \theta_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{v_r/c} \begin{pmatrix} 0 & -\sin \theta_r & 0 \\ \sin \theta_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similar relations hold for the other principal rotation axes with change of base, as well as in the derived polar bases. I bet that it will be possible to prove all of physics with (the calculus of) **expk**, **expj**, and **expi** (in an electromagnetic basis of heat).

Also importantly, given that the Fresnel equations are obtained as ratios of complex exponentials assuming that electromagnetic fields are complex exponentials (and hence the ratios, reflected and transmitted over incident), converting this theory over to basis (quaternion) exponentials involves divisions... one exponential divided by another. Very importantly and pleasing is that basis (quaternion) exponentials (**expk**, **expj**, and **expi**) are full rank so there are no issues here with these divisions.

Not surprisingly, it all works for principal axis theorems for moments of inertia which I had been intending to write about. Mohr's circle for stresses and strains from the mechanics of solids as well, which is engineering physics (for bridges, buildings, and skyscrapers).

By defining $i = \hat{k} \times$ [that is, as a vector rotation operator] all of the elementary functions have generalized into vector functions in R^3 and this means all of traditional classical excellence (traditional Oxford and Cambridge calculus) extends also to vector functions in R^3 which I am, understandably, very pleased about.